

Curves

(PARI-GP version 2.19.0)

Conics

find a rational point on a conic, ${}^t x G x = 0$ `qfsolve(G)`
[*H*, *U*] such that $H = c^t U G U$ has minimat `qfminimize(G)`
quadratic Hilbert symbol (at *p*) `hilbert(x, y, {p})`
all solutions in $\mathbf{Q}^3$ of ternary form `qfparam(G, x)`

Elliptic curves

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain *D* `E = ellinit(v, {D = 1})`
over $\mathbf{Q}$ *D* = 1
over $\mathbf{F}_p$ *D* = *p*
over $\mathbf{F}_q$, $q = p^f$ *D* = `ffgen`([*p*, *f*])
over $\mathbf{Q}_p$, precision *n* *D* = $O(p^n)$
over $\mathbf{C}$, current bitprecision *D* = 1.0
over number field *K* *D* = *nf*

Points are [*x*,*y*], the origin is [0]. Struct members:

- All domains: **E.a1,a2,a3,a4,a6**, **b2,b4,b6,b8**, **c4,c6**, **disc**, **j**
- *E* defined over **R** or **C**
x-coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega**, **E.eta**
volume of complex lattice **E.area**
- *E* defined over $\mathbf{Q}_p$
residual characteristic **E.p**
If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- *E* defined over $\mathbf{F}_q$
characteristic **E.p**
 $\#E(\mathbf{F}_q)$ /cyclic structure/generators **E.no**, **E.cyc**, **E.gen**
- *E* defined over $\mathbf{Q}$
generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from *j*-invariant **ellfromj**(*j*)
cubic/quartic/biquadratic to Weierstrass **ellfromeqn**(*eq*)
add points $P + Q$ / $P - Q$ **elladd**(*E*, *P*, *Q*), **ellsub**
negate point **ellneg**(*E*, *P*)
compute $n \cdot P$ **ellmul**(*E*, *P*, *n*)
sum of Galois conjugates of *P* **elltrace**(*E*, *P*)
check if *P* is on *E* **ellisoncurve**(*E*, *P*)
order of torsion point *P* **ellorder**(*E*, *P*)
y-coordinates of point(*s*) for *x* **ellordinate**(*E*, *x*)
 $[\wp(z), \wp'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ **ellztpoint**(*E*, *z*)
 $z \in \mathbf{C}$ such that $P = [\wp(z), \wp'(z)]$ **ellpointtoz**(*E*, *P*)
 $z \in \tilde{\mathbf{Q}}^*/q^{\mathbf{Z}}$ to $P \in E(\tilde{\mathbf{Q}}_p)$ **ellztopoint**(*E*, *z*)
 $P \in E(\tilde{\mathbf{Q}}_p)$ to $z \in \tilde{\mathbf{Q}}^*/q^{\mathbf{Z}}$ **ellpointtoz**(*E*, *P*)
- Change of Weierstrass models**, using $v = [u, r, s, t]$
compose/invert changes **ellchangecompose**, **ellchangeinvert**
change curve *E* using *v* **ellchangecurve**(*E*, *v*)
change point *P* using *v* **ellchangepoint**(*P*, *v*)
change point *P* using inverse of *v* **ellchangepointinv**(*P*, *v*)
is *E* isomorphic to *F*? **ellsisom**(*E*, *F*)

Twists and isogenies

quadratic twist **elltwtst**(*E*, *d*)
n-division polynomial $f_n(x)$ **elldivpol**(*E*, *n*, {*x*})
 $[n]P = (\phi_n \psi_n; \omega_n; \psi_n^3)$; return (ϕ_n, ψ_n^2) **ellxn**(*E*, *n*, {*x*})
isogeny from *E* to *E*/*G* **ellisogeny**(*E*, *G*)
apply isogeny to *g* (point or isogeny) **ellisogenyapply**(*f*, *g*)
torsion subgroup with generators **elltors**(*E*)

Formal group

formal exponential, *n* terms **ellformalexp**(*E*, {*n*}, {*x*})
formal logarithm, *n* terms **ellformallog**(*E*, {*n*}, {*x*})
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog**(*E*, *p*, *n*, *P*)
P in the formal group **ellformalpoint**(*E*, {*n*}, {*x*})
 $[\omega/dt, x\omega/dt]$ **ellformaldifferential**(*E*, {*n*}, {*x*})
 $w = -1/y$ in parameter $-x/y$ **ellformalw**(*E*, {*n*}, {*x*})

Elliptic curves over finite fields, Pairings

random point on *E* **random**(*E*)
 $\#E(\mathbf{F}_q)$ **ellcard**(*E*)
 $\#E(\mathbf{F}_q)$ with almost prime order **ellsea**(*E*, {*tors*})
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup**(*E*)
is *E* supersingular? **ellissupersingular**(*E*)
random supersingular *j*-invariant over F_p^2 **ellsupersingularj**(*p*)
Weil pairing of *m*-torsion pts *P*, *Q* **ellweilpairing**(*E*, *P*, *Q*, *m*)
Tate pairing of *P*, *Q*; *P* *m*-torsion **elltatepairing**(*E*, *P*, *Q*, *m*)
Discrete log, find *n* s.t. $P = [n]Q$ **elllog**(*E*, *P*, *Q*, {*ord*})

Elliptic curves over Q

Reduction, minimal model

minimal model of *E*/ $\mathbf{Q}$ **ellminimalmodel**(*E*, {*fv*})
quadratic twist of minimal conductor **ellminimaltwist**(*E*)
 $[k]P$ with good reduction **ellnonsingularmultiple**(*E*, *P*)
E supersingular at *p*? **ellissupersingular**(*E*, *p*)
affine points of naïve height $\leq h$ **ellratpoints**(*E*, *h*)

Complex heights

canonical height of *P* **ellheight**(*E*, *P*)
canonical bilinear form taken at *P*, *Q* **ellheight**(*E*, *P*, *Q*)
height regulator matrix for pts in *L* **ellheightmatrix**(*E*, *L*)

p-adic heights

cyclotomic *p*-adic height of $P \in E(\mathbf{Q})$ **ellpadicheight**(*E*, *p*, *n*, *P*)
... bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight**(*E*, *p*, *n*, *P*, *Q*)
... matrix at vector for pts in *L* **ellpadicheightmatrix**(*E*, *p*, *n*, *L*)
... regulator for canonical height **ellpadicregulator**(*E*, *p*, *n*, *Q*)
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobenius**(*E*, *p*, *n*)
slope of unit eigenvector of Frobenius **ellpadics2**(*E*, *p*, *n*)

Isogenous curves

matrix of isogeny degrees for $\mathbf{Q}$ -isog. curves **ellisomat**(*E*)
tree of prime degree isogenies **ellisotree**(*E*)
a modular equation of prime degree *N* **ellmodulareqn**(*N*)

L-function

char. poly. of Frobenius **ellcharpoly**(*E*, *p*)
p-th coeff a_p of *L*-function, *p* prime **ellap**(*E*, *p*)
k-th coeff a_k of *L*-function **ellak**(*E*, *k*)
L(*E*, *s*) (using less memory than **lfun**) **elllseries**(*E*, *s*)
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **ellL1**(*E*, *r*)
order of vanishing at 1 **ellanalyticrank**(*E*, {*eps*})
root number for $L(E, \cdot)$ at *p* **ellrootno**(*E*, {*p*})
modular parametrization of *E* **elltaniyama**(*E*)
degree of modular parametrization **ellmoddegree**(*E*)
compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) **ellweilcurve**(*E*)
Manin constant of *E* **ellmaninconstant**(*E*)

p-adic *L* function $L_p^{(r)}(E, d, \chi^s)$ **ellpadicL**(*E*, *p*, *n*, {*s*}, {*r*}, {*d*})
BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ **ellpadicbsd**(*E*, *p*, *n*, {*D* = 1})
Iwasawa invariants for $L_p(E_D, \tau^i)$ **ellpadiclambdamu**(*E*, *p*, *D*, *i*)

Rational points

a Heegner point on *E* of rank 1 **ellheegner**(*E*)
... on the twist E_D (of rank 1) **ellheegnertwist**(*E*, *D*)
attempt to compute $E(\mathbf{Q})$ **ellrank**(*E*, {*effort*}, {*points*})
initialize for later **ellrank** calls, **ellrankinit**(*E*)
saturate $(P_1, \dots, P_n)$ wrt. primes $\leq B$ **ellsaturation**(*E*, *P*, *B*)
2-covers of the curve *E* **ell2cover**(*E*)

Elldata package, Cremona's database:

db code "11a1" $\leftrightarrow$ [*conductor*, *class*, *index*] **ellconvertname**(*s*)
generators of Mordell-Weil group **ellgenerators**(*E*)
look up *E* in database **ellidentify**(*E*)
all curves matching criterion **ellsearch**(*N*)
loop over curves with cond. from *a* to *b* **forell**(*E*, *a*, *b*, *seq*)

Elliptic curves over number field K

coeff a_p of *L*-function **ellap**(*E*, *p*)
Kodaira type of $\mathfrak{p}$ -fiber of *E* **elllocalred**(*E*, *p*)
integral model of *E*/*K* **ellintegralmodel**(*E*, {*fv*})
minimal model of *E*/*K* **ellminimalmodel**(*E*, {*fv*})
minimal discriminant of *E*/*K* **ellminimaldisc**(*E*)
cond, min mod, Tamagawa num [*N*, *v*, *c*] **ellglobalred**(*E*)
global Tamagawa number **elltamagawa**(*E*)
test if *E* has CM **elliscm**(*E*)
 $P \in E(K)$ *n*-divisible? $[n]Q = P$ **ellisdivisible**(*E*, *P*, *n*, {*Q*})

L-function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height *h*).
vector of first *n* a_k 's in *L*-function **ellan**(*E*, *n*)
init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfunit**(*E*, *D*, {*n* = 0})
compute $L(E, s)$ (*n*-th derivative) **lfun**(*L*, *s*, {*n* = 0})
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD **ellbsd**(*E*)

Based on an earlier version by Joseph H. Silverman

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Hyperelliptic curves

A hyperelliptic curve C is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
check if $[x, y]$ is on C `hyperellisoncurve(C, [x, y])`
 y -coordinates of point(s) for x `hyperellordinate(C, x)`
Igusa invariants for C of genus 2 `genus2igusa(C)`
period matrix `hyperellperiods(C)`
affine rational points of height $\leq h$ `hyperellratpoints(C, h)`
Change of Weierstrass models, using $m = [e, [a, b, c, d], H]$
compose changes `hyperellchangecompose(C, m1, m2)`
invert change `hyperellchangeinvert(C, m)`
change curve C using m `hyperellchangecurve(C, m)`
change point P using m `hyperellchangepoint(P, m)`
change point P using inverse of m `hyperellchangepointinv(P, v)`
automorphisms of C `hyperellauto(C, {nf})`
is C_1 isomorphic to C_2 ? `hyperellisom(C1, C2)`
Cremona-Stoll reduction `hyperellred(C)`
Reduction, minimal model
discriminant of C `hyperelldisc(C)`
minimal discriminant of integral C `hyperellminimaldisc(C)`
minimal model of integral C `hyperellminimalmodel(C)`
maximal chain of minimal models at `hyperellxtremalmodels(C, p)`
reduction of $y^2 + Qy = P$ (genus 2) `genus2red(C, {p})`
Frobenius
char. poly. of Frobenius `genus2charpoly(C, p)`
matrix of Frobenius on $\mathbf{Q}_p \otimes H^1_{dR}$ `hyperellpadicfrobenius`
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius `hyperellcharpoly(C)`

Curves

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Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or ell struct `(E.omega)`, $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean `agm(x, y)`
elliptic j -function $1/q + 744 + \dots$ `ellj(x)`
Weierstrass $\sigma/\wp/\zeta$ function `ellsigma(w, z), ellwp, ellzeta`
periods/quasi-periods `ellperiods(E, {flag}), elleta(w)`
 $(\omega_1, \omega_2), (g_2, g_3), (e_1, e_2, e_3), (\eta_1, \eta_2)$ `ellweierstrass(w)`
 $(2i\pi/\omega_2)^k E_k(\tau)$ `elleisnum(w, k)`
modified Dedekind η func. $\prod_n (1 - q^n)$ `eta(x)`
 $\dots$ true η function $q^{1/24} \prod_n (1 - q^n)$ `eta(x, 1)`
Dedekind sum $s(h, k)$ `sumdedekind(h, k)`
Riemann-Jacobi theta functions $\vartheta_{a,b}(z; \tau)$ `theta(z, t, [a, b])`
 $\dots$ Thetanullwerte (at $z = 0$) `thetanull(t, {[a, b]})`
Jacobi sine-theta function $\theta(q; z)$ `theta(q, z)`
 $\dots \theta^{(k)}(q, 0)$ `thetanullk(q, k)`
Jacobi elliptic functions (sn, cn, dn) `elljacobi(z, k)`
Weber's f functions `weber(x, {flag})`
modular pol. of level N `polmodular(N, {inv = j})`
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ `polclass(D, {inv = j})`