User's Guide

to

the PARI library

(version 2.19.0)

The PARI Group

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Table of Contents

Chapter 4: Programming PARI in Library Mode . . . 13
4.1 Introduction: initializations, universal objects . . . 13
4.2 Important technical notes . . . 14
4.2.1 Backward compatibility . . . 14
4.2.2 Types . . . 14
4.2.3 Type recursivity . . . 15
4.2.4 Variations on basic functions . . . 15
4.2.5 Portability: 32-bit / 64-bit architectures . . . 16
4.2.6 Using `malloc` / `free` . . . 16
4.3 Garbage collection . . . 17
4.3.1 Why and how . . . 17
4.3.2 Gory technical details . . . 19
4.3.3 The basic 3 garbage collecting functions . . . 19
4.3.4 A first detailed example . . . 20
4.3.5 More examples: variations on a theme . . . 21
4.3.6 Random garbage collection . . . 22
4.3.7 Final comments . . . 22
4.4 Creation of PARI objects, assignments, conversions . . . 23
4.4.1 Creation of PARI objects . . . 23
4.4.2 Sizes . . . 24
4.4.3 Assignment and copy . . . 25
4.4.4 Clones . . . 25
4.4.5 Conversions . . . 25
4.5 Implementation of the PARI types . . . 26
4.5.1 Type `t_INT` (integer) . . . 27
4.5.2 Type `t_REAL` (real number) . . . 28
4.5.3 Type `t_INTMOD` . . . 29
4.5.4 Type `t_FRAC` (rational number) . . . 29
4.5.5 Type `t_FFELT` (finite field element) . . . 29
4.5.6 Type `t_COMPLEX` (complex number) . . . 29
4.5.7 Type `t_PADIC` (p-adic numbers) . . . 29
4.5.8 Type `t_QUAD` (quadratic number) . . . 30
4.5.9 Type `t_POLMOD` (polmod) . . . 30
4.5.10 Type `t_POL` (polynomial) . . . 30
4.5.11 Type `t_SER` (power series) . . . 31
4.5.12 Type `t_RFRAC` (rational function) . . . 31
4.5.13 Type `t_QFB` (binary quadratic form) . . . 31
4.5.14 Type `t_VEC` and `t_COL` (vector) . . . 32
4.5.15 Type `t_MAT` (matrix) . . . 32
4.5.16 Type `t_VECSMALL` (vector of small integers) . . . 32
4.5.17 Type `t_STR` (character string) . . . 32
4.5.18 Type `t_ERROR` (error context) . . . 32
4.5.19 Type `t_CLOSURE` (closure) . . . 32
4.5.20 Type `t_INFINITY` (infinity) . . . 32
4.5.21 Type `t_LIST` (list) . . . 32
4.6 PARI variables . . . 33

4.6.1 Multivariate objects . . . 33
4.6.2 Creating variables . . . 33
4.6.3 Comparing variables . . . 35
4.7 Input and output . . . 35
4.7.1 Input . . . 35
4.7.2 Output to screen or file, output to string . . . 36
4.7.3 Errors . . . 38
4.7.4 Warnings . . . 38
4.7.5 Debugging output . . . 39
4.7.6 Timers and timing output . . . 39
4.8 Iterators, Numerical integration, Sums, Products . . . 40
4.8.1 Iterators . . . 40
4.8.2 Iterating over primes . . . 42
4.8.3 Parallel iterators . . . 43
4.8.4 parallel data processing . . . 45
4.8.5 Numerical analysis . . . 45
4.9 Catching exceptions . . . 46
4.9.1 Basic use . . . 46
4.9.2 Advanced use . . . 46
4.10 A complete program . . . 47
Chapter 5: Technical Reference Guide: the basics . . . 51
5.1 Initializing the library . . . 51
5.1.1 General purpose . . . 51
5.1.2 Technical functions . . . 52
5.1.3 Notions specific to the GP interpreter . . . 54
5.1.4 Public callbacks . . . 55
5.1.5 Configuration variables . . . 56
5.1.6 Utility functions . . . 56
5.1.7 Saving and restoring the GP context . . . 57
5.1.8 GP history . . . 57
5.2 Handling `GEN`s . . . 58
5.2.1 Allocation . . . 58
5.2.2 Length conversions . . . 59
5.2.3 Read type-dependent information . . . 60
5.2.4 Eval type-dependent information . . . 62
5.2.5 Set type-dependent information . . . 62
5.2.6 Type groups . . . 63
5.2.7 Accessors and components . . . 64
5.3 Global numerical constants . . . 64
5.3.1 Constants related to word size . . . 65
5.3.2 Masks used to implement the `GEN` type . . . 65
5.3.3 $\log 2$, π . . . 66
5.4 Iterating over small primes, low-level interface . . . 66
5.5 Handling the PARI stack . . . 67
5.5.1 Allocating memory on the stack . . . 67
5.5.2 Stack-independent binary objects . . . 68
5.5.3 Garbage collection . . . 68
5.5.4 Garbage collection: advanced use . . . 71
5.5.5 Debugging the PARI stack . . . 72

5.5.6 Copies . . . 72
5.5.7 Simplify . . . 72
5.6 The PARI heap . . . 72
5.6.1 Introduction . . . 72
5.6.2 Public interface . . . 73
5.6.3 Implementation note . . . 73
5.7 Handling user and temp variables . . . 74
5.7.1 Low-level . . . 74
5.7.2 User variables . . . 74
5.7.3 Temporary variables . . . 74
5.8 Adding functions to PARI . . . 75
5.8.1 Nota Bene . . . 75
5.8.2 Coding guidelines . . . 75
5.8.3 GP prototypes . . . 76
5.8.4 Integration with `gp` as a shared module . . . 78
5.8.5 Library interface for `install` . . . 79
5.8.6 Integration by patching `gp` . . . 79
5.9 Globals related to PARI configuration . . . 80
5.9.1 PARI version numbers . . . 80
5.9.2 Miscellaneous . . . 80
Chapter 6: Arithmetic kernel: Level 0 and 1 . . . 81
6.1 Level 0 kernel (operations on ulongs) . . . 81
6.1.1 Micro-kernel . . . 81
6.1.2 Modular kernel . . . 82
6.1.3 Modular kernel with "precomputed inverse" . . . 84
6.1.4 Switching between Fl_xxx and standard operators . . . 85
6.2 Level 1 kernel (operations on longs, integers and reals) . . . 86
6.2.1 Creation . . . 86
6.2.2 Assignment . . . 87
6.2.3 Copy . . . 87
6.2.4 Conversions . . . 88
6.2.5 Integer parts . . . 89
6.2.6 2-adic valuations and shifts . . . 90
6.2.7 From `t_INT` to bits or digits in base 2^k and back . . . 91
6.2.8 Integer valuation . . . 91
6.2.9 Generic unary operators . . . 93
6.2.10 Comparison operators . . . 93
6.2.11 Generic binary operators . . . 95
6.2.12 Exact division and divisibility . . . 98
6.2.13 Division with integral operands and `t_REAL` result . . . 98
6.2.14 Division with remainder . . . 98
6.2.15 Modulo to longs . . . 99
6.2.16 Powering, Square root . . . 100
6.2.17 GCD, extended GCD and LCM . . . 101
6.2.18 Continued fractions and convergents . . . 102
6.2.19 Pseudo-random integers . . . 102
6.2.20 Modular operations . . . 103
6.2.21 Extending functions to vector inputs . . . 106
6.2.22 Miscellaneous arithmetic functions . . . 106

Chapter 7: Level 2 kernel . . . 109
7.1 Naming scheme . . . 109
7.2 Coefficient ring . . . 111
7.3 Modular arithmetic . . . 112
7.3.1 FpC / FpV, FpM . . . 112
7.3.2 Flc / Flv, Flm . . . 116
7.3.3 F2c / F2v, F2m . . . 119
7.3.4 F3c / F3v, F3m . . . 121
7.3.5 FlxqV, FlxqC, FlxqM . . . 122
7.3.6 FpX . . . 123
7.3.7 FpXQ, Fq . . . 127
7.3.8 FpXQ . . . 129
7.3.9 Fq . . . 130
7.3.10 FpXn . . . 132
7.3.11 FpXC, FpXM . . . 132
7.3.12 FpXX, FpXY . . . 132
7.3.13 FpXk . . . 133
7.3.14 FpXQX, FqX . . . 133
7.3.15 FpXQXn, FqXn . . . 135
7.3.16 FpXQXQ, FqXQ . . . 136
7.3.17 Flx . . . 139
7.3.18 FlxV . . . 145
7.3.19 FlxM . . . 145
7.3.20 FlxT . . . 145
7.3.21 Flxn . . . 146
7.3.22 Flxq . . . 146
7.3.23 FlxX . . . 148
7.3.24 FlxXV, FlxXC, FlxXM . . . 149
7.3.25 FlxqX . . . 150
7.3.26 FlxqXQ . . . 153
7.3.27 FlxqXn . . . 154
7.3.28 F2x . . . 154
7.3.29 F2xq . . . 156
7.3.30 F2xn . . . 157
7.3.31 F2xqV, F2xqM . . . 157
7.3.32 F2xX . . . 158
7.3.33 F2xXV/F2xXC . . . 158
7.3.34 F2xqX . . . 158
7.3.35 F2xqXQ . . . 160
7.3.36 Functions returning objects with t_INTMOD coefficients . . . 160
7.3.37 Slow Chinese remainder theorem over **Z** . . . 162
7.3.38 Fast remainders . . . 163
7.3.39 Fast Chinese remainder theorem over **Z** . . . 164
7.3.40 Rational reconstruction . . . 165
7.3.41 Zp . . . 166
7.3.42 ZpM . . . 166
7.3.43 ZpX . . . 167
7.3.44 ZpXQ . . . 168
7.3.45 Zq . . . 168

7.3.46 ZpXQM . . . 168
7.3.47 ZpXQX . . . 169
7.3.48 ZqX . . . 169
7.3.49 ZpXQXQ . . . 169
7.3.50 Other p-adic functions . . . 170
7.3.51 Conversions involving single precision objects . . . 171
7.4 Higher arithmetic over **Z**: primes, factorization . . . 175
7.4.1 Pure powers . . . 175
7.4.2 Factorization . . . 176
7.4.3 Coprime factorization . . . 178
7.4.4 Checks attached to arithmetic functions . . . 178
7.4.5 Incremental integer factorization . . . 179
7.4.6 Integer core, squarefree factorization . . . 180
7.4.7 Primes, primality and compositeness tests . . . 180
7.4.8 Iterators over primes . . . 182
7.5 Integral, rational and generic linear algebra . . . 182
7.5.1 ZC / ZV, ZM . . . 182
7.5.2 QM . . . 187
7.5.3 Qevproj . . . 187
7.5.4 zv, zm . . . 188
7.5.5 ZMV / zmV (vectors of ZM/zm) . . . 189
7.5.6 QC / QV, QM . . . 189
7.5.7 RgC / RgV, RgM . . . 189
7.5.8 ZG . . . 195
7.5.9 Sparse and blackbox linear algebra . . . 196
7.5.10 Obsolete functions . . . 197
7.6 Integral, rational and generic polynomial arithmetic . . . 198
7.6.1 ZX . . . 198
7.6.2 Resultants . . . 201
7.6.3 ZXV . . . 201
7.6.4 ZXT . . . 201
7.6.5 ZXQ . . . 202
7.6.6 ZXn . . . 202
7.6.7 ZXQM . . . 202
7.6.8 ZXQX . . . 202
7.6.9 ZXX . . . 203
7.6.10 ZXk/QXk . . . 203
7.6.11 QX . . . 204
7.6.12 QXQ . . . 204
7.6.13 QXQX . . . 205
7.6.14 QXQM . . . 205
7.6.15 zx . . . 206
7.6.16 RgX . . . 206
7.6.17 RgXn . . . 212
7.6.18 RgXnV . . . 213
7.6.19 RgXQ . . . 213
7.6.20 RgXQV, RgXQC . . . 214
7.6.21 RgXQM . . . 214
7.6.22 RgXQX . . . 214

Chapter 8: Black box algebraic structures . . . 215
8.1 Black box groups . . . 215
8.1.1 Black box groups with pairing . . . 217
8.1.2 Functions returning black box groups . . . 217
8.2 Black box fields . . . 218
8.2.1 Functions returning black box fields . . . 219
8.3 Black box algebra . . . 219
8.3.1 Functions returning black box algebras . . . 220
8.4 Black box ring . . . 220
8.5 Black box free **Z**_*p*-modules . . . 221
Chapter 9: Operations on general PARI objects . . . 223
9.1 Conversions . . . 223
9.1.1 Scalars . . . 223
9.1.2 Modular objects / lifts . . . 225
9.1.3 Between polynomials and coefficient arrays . . . 225
9.2 Constructors . . . 227
9.2.1 Clean constructors . . . 227
9.2.2 Unclean constructors . . . 230
9.2.3 From roots to polynomials . . . 234
9.3 Integer parts . . . 234
9.4 Valuation and shift . . . 234
9.5 Comparison operators . . . 235
9.5.1 Generic . . . 235
9.5.2 Comparison with a small integer . . . 235
9.6 Miscellaneous Boolean functions . . . 236
9.6.1 Obsolete . . . 237
9.7 Sorting . . . 237
9.7.1 Basic sort . . . 237
9.7.2 Indirect sorting . . . 237
9.7.3 Generic sort and search . . . 238
9.7.4 Further useful comparison functions . . . 239
9.8 Division . . . 239
9.9 Divisibility, Euclidean division . . . 240
9.10 GCD, content and primitive part . . . 241
9.10.1 Generic . . . 241
9.10.2 Over the rationals . . . 241
9.11 Generic arithmetic operators . . . 243
9.11.1 Unary operators . . . 243
9.11.2 Binary operators . . . 243
9.12 Generic operators: product, powering, factorback . . . 244
9.13 Matrix and polynomial norms . . . 245
9.14 Substitution and evaluation . . . 246
Chapter 10: Miscellaneous mathematical functions . . . 249
10.1 Fractions . . . 249
10.2 Binomials . . . 249
10.3 Real numbers . . . 249
10.4 Complex numbers . . . 250
10.5 Low accuracy input or output . . . 250
10.6 Quadratic numbers and binary quadratic forms . . . 251

10.7 Polynomials . . . 251
10.7.1 Obsolete functions . . . 252
10.8 Power series . . . 252
10.9 Functions to handle t_FFELT . . . 253
10.9.1 FFX . . . 256
10.9.2 FFM . . . 257
10.9.3 FFXQ . . . 258
10.10 Transcendental functions . . . 258
10.10.1 Transcendental functions with t_REAL arguments . . . 258
10.10.2 Other complex transcendental functions . . . 260
10.10.3 Modular functions . . . 261
10.10.4 Transcendental functions with t_PADIC arguments . . . 262
10.10.5 Cached constants . . . 262
10.10.6 Obsolete functions . . . 263
10.11 Permutations . . . 263
10.12 Small groups . . . 264
Chapter 11: Standard data structures . . . 269
11.1 Character strings . . . 269
11.1.1 Functions returning a char * . . . 269
11.1.2 Functions returning a t_STR . . . 270
11.1.3 Dynamic strings . . . 270
11.2 Output . . . 271
11.2.1 Output contexts . . . 271
11.2.2 Default output context . . . 271
11.2.3 PARI colors . . . 272
11.2.4 Obsolete output functions . . . 272
11.3 Files . . . 273
11.3.1 pariFILE . . . 273
11.3.2 Temporary files . . . 274
11.4 Errors . . . 274
11.4.1 Internal errors, “system” errors . . . 274
11.4.2 Syntax errors, type errors . . . 275
11.4.3 Overflows . . . 276
11.4.4 Errors triggered intentionally . . . 277
11.4.5 Mathematical errors . . . 278
11.4.6 Miscellaneous functions . . . 279
11.5 Hashtables . . . 279
11.6 Dynamic arrays . . . 282
11.6.1 Initialization . . . 282
11.6.2 Adding elements . . . 282
11.6.3 Accessing elements . . . 282
11.6.4 Stack of stacks . . . 282
11.6.5 Public interface . . . 283
11.7 Vectors and Matrices . . . 283
11.7.1 Access and extract . . . 283
11.7.2 Componentwise operations . . . 285
11.7.3 Low-level vectors and columns functions . . . 285
11.8 Vectors of small integers . . . 286
11.8.1 t_VECSMALL . . . 286

11.8.2 Vectors of t_VECSMALL . . . 288
Chapter 12: Functions related to the GP interpreter . . . 289
12.1 Handling closures . . . 289
12.1.1 Functions to evaluate t_CLOSURE . . . 289
12.1.2 Control flow functions . . . 290
12.1.3 Functions to handle control flow changes . . . 291
12.1.4 Functions to deal with lexical local variables . . . 291
12.1.5 Functions returning new closures . . . 292
12.1.6 Functions used by the gp debugger (break loop) . . . 292
12.1.7 Standard wrappers for iterators . . . 292
12.2 Defaults . . . 293
12.3 Records and Lazy vectors . . . 296
Chapter 13: Algebraic Number Theory . . . 299
13.1 General Number Fields . . . 299
13.1.1 Number field types . . . 299
13.1.2 Extracting info from a nf structure . . . 301
13.1.3 Extracting info from a bnf structure . . . 302
13.1.4 Extracting info from a bnr structure . . . 303
13.1.5 Extracting info from an rnf structure . . . 304
13.1.6 Extracting info from a bid structure . . . 305
13.1.7 Extracting info from a znstar structure . . . 306
13.1.8 Inserting info in a number field structure . . . 306
13.1.9 Increasing accuracy . . . 307
13.1.10 Number field arithmetic . . . 307
13.1.11 Number field arithmetic for linear algebra . . . 310
13.1.12 Cyclotomic field arithmetic for linear algebra . . . 311
13.1.13 Cyclotomic trace . . . 311
13.1.14 Elements in factored form . . . 311
13.1.15 Ideal arithmetic . . . 313
13.1.16 Maximal ideals . . . 316
13.1.17 Decomposition groups . . . 318
13.1.18 Reducing modulo maximal ideals . . . 318
13.1.19 Valuations . . . 319
13.1.20 Signatures . . . 320
13.1.21 Complex embeddings . . . 321
13.1.22 Maximal order and discriminant, conversion to nf structure . . . 322
13.1.23 Computing in the class group . . . 323
13.1.24 Floating point embeddings, the T_2 quadratic form . . . 325
13.1.25 Ideal reduction, low level . . . 325
13.1.26 Ideal reduction, high level . . . 326
13.1.27 Class field theory . . . 327
13.1.28 Abelian maps . . . 329
13.1.29 Grunwald–Wang theorem . . . 330
13.1.30 Relative equations, Galois conjugates . . . 330
13.1.31 Units . . . 332
13.1.32 Obsolete routines . . . 332
13.2 Galois extensions of $\mathbf{Q}$ . . . 333
13.2.1 Extracting info from a gal structure . . . 333
13.2.2 Miscellaneous functions . . . 334

13.3 Quadratic number fields and quadratic forms . . . 334
13.3.1 Checks . . . 334
13.3.2 Class number . . . 334
13.3.3 t_QFB . . . 335
13.3.4 Efficient real quadratic forms . . . 337
13.4 Linear algebra over **Z** . . . 339
13.4.1 Hermite and Smith Normal Forms . . . 339
13.4.2 The LLL algorithm . . . 343
13.4.3 Linear dependencies . . . 345
13.4.4 Reduction modulo matrices . . . 345
13.5 Finite abelian groups and characters . . . 346
13.5.1 Abstract groups . . . 346
13.5.2 Dirichlet characters . . . 347
13.6 Hecke characters . . . 348
13.7 Central simple algebras . . . 349
13.7.1 Initialization . . . 349
13.7.2 Type checks . . . 349
13.7.3 Shallow accessors . . . 350
13.7.4 Other low-level functions . . . 351
Chapter 14: Elliptic curves and arithmetic geometry . . . 353
14.1 Elliptic curves . . . 353
14.1.1 Types of elliptic curves . . . 353
14.1.2 Type checking . . . 353
14.1.3 Extracting info from an ell structure . . . 354
14.1.4 Points . . . 358
14.1.5 Change of variables . . . 358
14.1.6 Generic helper functions . . . 358
14.1.7 Functions to handle elliptic curves over finite fields . . . 359
14.2 Arithmetic on elliptic curve over a finite field in simple form . . . 359
14.2.1 Helper functions . . . 359
14.2.2 Elliptic curves over $\mathbf{F}_p$, $p > 3$ . . . 360
14.2.3 FpE . . . 360
14.2.4 Fle . . . 361
14.2.5 FpJ . . . 362
14.2.6 Flj . . . 363
14.2.7 Elliptic curves over $\mathbf{F}_{2^n}$ . . . 363
14.2.8 F2xqE . . . 363
14.2.9 Elliptic curves over $\mathbf{F}_q$, small characteristic $p > 2$ . . . 364
14.2.10 FlxqE . . . 364
14.2.11 Elliptic curves over $\mathbf{F}_q$, large characteristic . . . 365
14.2.12 FpXQE . . . 366
14.3 Functions related to modular polynomials . . . 367
14.3.1 Functions related to modular invariants . . . 367
14.4 Other curves . . . 367
Chapter 15: L-functions . . . 369
15.1 Accessors . . . 369
15.2 Conversions and constructors . . . 370
15.3 Variants of GP functions . . . 371
15.4 Inverse Mellin transforms of Gamma products . . . 371

Chapter 16: Modular symbols 373
Chapter 17: Modular forms 375
17.1 Implementation of public data structures 375
17.1.1 Accessors for modular form spaces 375
17.1.2 Accessors for individual modular forms 376
17.1.3 Nebentypus 377
17.1.4 Miscellaneous functions 377
Chapter 18: Plots 379
18.1 Highlevel functions 379
18.2 Function 380
18.2.1 Obsolete functions 380
18.3 Dump rectwindows to a PostScript or SVG file 381
18.4 Technical functions exported for convenience 381
Appendix A: A Sample program and Makefile 383
Appendix B: PARI and threads 385
Index 388

Chapter 4:
Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and describes high level PARI routines available through the calculator. The present manual focuses on the usage of the PARI library, assuming general familiarity with the contents of these chapters and with ANSI C programming. In the present chapter, we introduce concepts and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type **GEN**, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime` $= 0$, although the system still needs to precompute all primes up to about 2^{16}. For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

• a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

• the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1, −1, 2 and −2 (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction $\frac{1}{2}$ (`ghalf`). All of these are of type `GEN`.

• a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

• a table of primes, called `pari_PRIMES`, of type `pari_prime*` (pointer to 32bit unsigned integer). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

• access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

4.2 Important technical notes.

4.2.1 Backward compatibility. The PARI function names evolve over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the function prototype remained the same).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un`*: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its "value" will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

* For `(long)gen_1`. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the `GEN` type is a pointer to `long`, not to `GEN`. So special macros must be used to access `GEN`'s components. The simplest one is `gel`(V, i), where **el** stands for **el**ement, to access component number i of the `GEN` V. This is a valid `lvalue` (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;
x = gel(V, i);
```

where `x` and `V` are `GEN`s. This macro accesses and modifies directly the components of V and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the `gmael` macros (for **m**ultidimensional **a**rray **el**ement). The syntax is **gmael**$n(V, a_1, \ldots, a_n)$, where V is a `GEN`, the a_i are indexes, and n is an integer between 1 and 5. This stands for $x[a_1][a_2]\ldots[a_n]$, and returns a `GEN`. The macros `gel` (resp. `gmael`) are synonyms for `gmael1` (resp. `gmael2`).

Finally, the macro `gcoeff`(M, i, j) has exactly the meaning of `M[i,j]` in GP when `M` is a matrix. Note that due to the implementation of `t_MAT`s as horizontal lists of vertical vectors, `gcoeff(x,y)` is actually equivalent to `gmael(y,x)`. One should use `gcoeff` in matrix context, and `gmael` otherwise.

4.2.4 Variations on basic functions. In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example `gadd`(x, y) assumes that x and y are `GEN`s, and creates the result $x+y$ on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for `gadd`, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

```
GEN gaddgs(GEN x, long y)
```

```
GEN gaddsg(long x, GEN y)
```

Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (`long`s and `ulong`s), Level 1 with `t_INT` and `t_REAL` and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

`GEN addii(GEN x, GEN y)`: here x and y are `GEN`s of type `t_INT` (this is not checked).

`GEN addrr(GEN x, GEN y)`: here x and y are `GEN`s of type `t_REAL` (this is not checked).

There also exist functions **addir**, **addri**, **mpadd** (whose two arguments can be of type `t_INT` or `t_REAL`), **addis** (to add a `t_INT` and a `long`) and so on.

The Level 1 names are self-explanatory once you know that **i** stands for a `t_INT`, **r** for a `t_REAL`, **mp** for i or r, **s** for a signed C long integer, **u** for an unsigned C long integer. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of x and y modulo m. Here x, y, m are `t_INT`s (this is not checked). The operation is more efficient if the inputs x, y are reduced modulo m, but this is not a necessary condition.

Important Note. These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1. There, it is replaced by the simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also C `long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT` `gen_0` if the integer is 0.

4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

`DEFAULTPREC`	64	($\approx$ 19 decimal digits)
`MEDDEFAULTPREC`	128	($\approx$ 38 decimal digits)
`BIGDEFAULTPREC`	192	($\approx$ 57 decimal digits)

For instance, suppose you call a transcendental function, such as

`GEN` **`gexp`**`(GEN x, long prec)`.

The last argument `prec` is the default floating point precision required, given in bits. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`.

4.2.6 Using `malloc / free`. You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

`void* pari_malloc(size_t size)` calls `malloc` to allocate `size` bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The `SIGINT` signal is blocked until `malloc` returns, to avoid leaving the system stack in an inconsistent state.

`void* pari_realloc(void* ptr, size_t size)` as `pari_malloc` but calls `realloc` instead of `malloc`.

`void pari_realloc_ip(void** ptr, size_t size)` equivalent to `*ptr= realloc(*ptr, size)`, while blocking `SIGINT`.

`void* pari_calloc(size_t size)` as `pari_malloc`, setting the memory to zero.

`void pari_free(void* ptr)` calls `free` to liberate the memory space pointed to by `ptr`, which must have been allocated by `malloc` (`pari_malloc`) or `realloc` (`pari_realloc`). The `SIGINT` signal is blocked until `free` returns.

If you use the standard `libc` functions instead of our wrappers, then your functions will be subtly incompatible with the `gp` calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

4.3 Garbage collection.

4.3.1 Why and how.

As we have seen, `pari_init` allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address `bot` and ends just before `top`. This means that the quantity

$$(\texttt{top} - \texttt{bot}) / \texttt{sizeof(long)}$$

is (roughly) equal to the `size` argument of `pari_init`. The PARI stack also has a "current stack pointer" called **avma**, which stands for **av**ailable **m**emory **a**ddress. These three variables are global (declared by `pari.h`). They are of type `pari_sp`, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma` = `top`, and `avma` gets decremented as new objects are created. As its name indicates, `avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called "*garbage collecting*."

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsections.

• First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (undocumented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

• It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```
pari_sp av = avma; /* record initial avma */
... garbage ...
set_avma(av); /* restore it */
```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored. Think of the `set_avma` call as a simple `avma = av` restoring the `avma` value.

• If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

• If on the contrary, you want to keep the latest computed object z and destroy everything else since the reference point, use

```
pari_sp av = avma; /* record initial avma */
... garbage ...
oldz = fun();
newz = gc_upto(av, oldz); /* destroy garbage but keep z safe */
```

Of course, this can be written directly as

```
z = gc_upto(av, fun());
```

This form is actually preferred since it guarantees garbage collection occurs at the right time, just after `fun()` returns its result. Note that the `gc_upto` call moves `oldz` and in fact *destroys* it. One should never refer to it again, but use the updated `newz` instead.

• When an arbitrary `GEN` must be recovered from the middle of a computation, even if it was constructed incrementally, then you can use the most robust (hence slowest) garbage collection function: `gc_GEN`.

```
pari_sp av = avma; /* record initial avma */
... garbage ...
oldz = fun();
... garbage ...
newz = gc_GEN(av, oldz); /* destroy garbage but keep z safe */
```

This function even allows z to contain shallow data (pointers to other parts of the stack), which is forbidden for high-level PARI functions. Robustness comes at a price but this is at most twice slower than `gc_upto`. Again `oldz` should not be accessed anymore.

• Finally, when more that one result needs to be saved, you can use the following idiom:

```
pari_sp av = avma; /* record initial avma */
... garbage ...
z1 = fun();
... garbage ...
z2 = fun();
... garbage ...
gc_all(av, 2, &z1, &z2); /* destroy garbage, keep z1 and z2 safe */
```

As usual, `gc_all` changes `z1` and `z2`: the old values should no longer be accessed. This is a variadic function, which essentially performs as many simultaneous `gc_GEN` as needed: if n objects need to be updated, use

```
gc_all(av, n, &z1, ..., &zn);
```

4.3.2 Gory technical details.

You may skip this section on first reading. We will now describe the underlying mechanisms in terms of the lowest level function

`size_t gc_stack_update(pari_sp ltop, pari_sp lbot)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot` $\leq$ `ltop`, and returns the offset $o =$ `ltop` $-$ `lbot`. This means the following.

1) we translate (copy) all data in the interval [`avma`, `lbot`[to [`avma` $+ o$, `lbot` $+ o$]. Note that the right extremity is now exactly `ltop`. Graphically

```
              bot           avma   lbot          ltop     top
End of stack  |-------------[++++++[-/-/-/-/-/-/-|++++++++|  Start
                free memory            garbage
```

becomes:

```
              bot                          avma   ltop     top
End of stack  |----------------------------[++++++[++++++++|  Start
                          free memory
```

where `++` denote significant objects, `--` the unused part of the stack, and `-/-` the garbage we remove.

2) The function then inspects all the PARI objects between (the original) `avma` and `lbot`, i.e. the ones that we want to keep and that have been translated. Every component of such an object which is not a codeword is a pointer to an address which is either

– between `avma` and `lbot`, in which case we add to it the offset o;

– larger than or equal to `ltop`, in which case it does not change, or

– between `lbot` and `ltop` in which case the function raises an error: "significant pointers lost in gc".

3) **avma** is replaced by **avma** $+ o$ as in the diagram above.

4) The function returns the offset o.

As stated above, no component of the remaining objects should belong to the erased segment [`lbot`, `ltop`[, and the function checks this. But beware as well that the addresses of the objects in the translated zone have now changed, so you must not access any pointer which previously pointed into the zone below `ltop`.

4.3.3 The basic 3 garbage collecting functions.

Let us recap: although all is implemented in terms of the `gc_stack_update` low level function, you don't want to use it directly. You mostly want to use one of the following three functions to collect garbage (reclaim memory) while keeping some results safe. Many others will be described in Section 5.5.3 for specialized scenarios.

`GEN gc_upto(pari_sp ltop, GEN q)`. This applies when `q` is the last object on the stack, returned by a high level PARI function. It cleans the stack between `ltop` and the object `q` and returns `q` updated. Your own functions or low level PARI functions (such as shallow functions) described in the next Chapter may not satisfy this: this will be mentioned in their documentation.

`GEN gc_GEN(pari_sp ltop, GEN q)`. Works in all cases. Functionally equivalent to, but more efficient than

```
gc_upto(ltop, gcopy(q))
```

In this case, the `GEN` parameter `q` need not satisfy any property before the garbage collection. Of course, this is about twice slower than `gc_upto`, because `q` has to be copied to a clean stack zone first. This function is a special case of `gc_all` below, where $n = 1$.

`GEN gc_all(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects n further arguments, which are the *addresses* of the `GEN`s you want to preserve:

```
pari_sp ltop = avma;
garbage(); x = anything();
garbage(); y = anything();
garbage(); gc_all(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the `GEN`s added to the argument list. This is as robust as `gc_GEN`. For convenience, this function also returns the new value of x, the first object recovered.

4.3.4 A first detailed example.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute $\mathtt{x}^2 + \mathtt{y}^2$. This is done using the following program:

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The `GEN z` indeed points at the desired quantity. However, the stack contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `x2`, `y2`, `z`.

It is not possible to get rid of `x2`, `y2` before `z` is computed, since they are used in the final operation. We cannot record `avma` before `x2` is computed and restore it later, since this would destroy `z` as well. It is not possible either to use the function `cgiv` since `x2` and `y2` are not the latest computed objects and we do not want to give back `z`.

Using `gc_upto`, we can give back the memory locations corresponding to `x2`, `y2`, and move the object `z` upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma;  /* remember the current top of the stack */
GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
GEN z = gc_upto(ltop, gadd(x2, y2));
```

or even

```
ltop = avma;
z = gc_upto(ltop, gadd(gsqr(x), gsqr(y)));
```

This works because the output of `gadd` is the last object on the stack and everything else that was introduced after recording the `ltop` reference point is deleted. Another possibility is to use

```
ltop = avma;
z = gc_GEN(ltop, gadd(gsqr(x), gsqr(y)));
```

which would work in all cases, even if `gadd` were shallow. Garbage collection would be at most twice slower, and certainly negligible compared to the squaring times. Make sure you understand exactly what has happened before you go on!

4.3.5 More examples: variations on a theme. it is often necessary to do several garbage collections during a computation. However, the fewer the better. The only condition for `gc_upto` to work is that the garbage be connected and we need only recover the last result on the stack. The computation should be arranged so that there is a minimal number of connected pieces of garbage.

For example suppose we want to write a function of two `GEN` variables `x` and `y` which creates the vector $[x^2 + y, y^2 + x]$. Without garbage collecting, one would write:

```
x2 = gsqr(x); a = gadd(x2, y);
y2 = gsqr(y); b = gadd(y2, x);
z = mkvec2(a, b);  /* shallow: not suitable for gc_upto! */
```

This leaves a dirty stack containing `x2`, `a`, `y2`, `b` and `z`. The garbage here consists of `x2` and `y2`, which are separated by `a`, which we need to keep. But if we compute `y2` *before* `a` then the garbage becomes connected, and we get the following program with garbage collecting:

```
ltop = avma; x2 = gsqr(x); y2 = gsqr(y);
z = cgetg(3, t_VEC);
gel(z, 1) = gadd(x2,y);
gel(z, 2) = gadd(y2,x); z = gc_upto(ltop, z);
```

Beware that

```
ltop = avma; a = gadd(gsqr(x), y); b = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
gel(z, 1) = a;
gel(z, 2) = b; z = gc_upto(ltop,z); /* WRONG */
```

is a disaster since `a` and `b` are created before `z`, so the call to `gc_upto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data!

On the other hand

```
ltop = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(gsqr(x), y);
gel(z, 2) = gadd(gsqr(y), x); z = gc_upto(ltop,z); /* inefficient */
```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gc_upto` update them for naught). In some sense, the best version is

```
z = cgetg(3, t_VEC);
ltop = avma; gel(z, 1) = gc_upto(ltop, gadd(gsqr(x), y));
ltop = avma; gel(z, 2) = gc_upto(ltop, gadd(gsqr(y), x));
```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

One last example. Let us compute the product of two complex numbers x and y, using the $3M$ method which requires 3 multiplications instead of the obvious 4. Let $z = x*y$, and set $x = x_r+i*x_i$ and similarly for y and z. We compute $a = x_r * y_r$, $b = x_i * y_i$, $c = (x_r + x_i) * (y_r + y_i)$, and then we have $z_r = a - b$, $z_i = c - (a + b)$. The program is as follows:

```
ltop = avma;
a = gmul(gel(x,1), gel(y,1));
b = gmul(gel(x,2), gel(y,2));
c = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
d = gadd(a,b); z = cgetg(3, t_COMPLEX);
gel(z, 1) = gsub(a,b);
gel(z, 2) = gsub(c,d); z = gc_upto(ltop, z);
```

4.3.6 Random garbage collection.

The most important usage of `gc_all` is to make sure loops don't fill up the stack:

```
pari_sp av = avma;
GEN x, y;
while (...)
{
  garbage(); x = anything();
  garbage(); y = anything();
  garbage(); gc_all(av, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it can be costly to call `gc_all` once for each iteration, we may do this only when it is necessary:

```
/* memory is running low (half spent since entry) */
if (gc_needed(av,1)) gc_all(av, 2, &x, &y);
```

More precisely, the macro `stack_lim(av,`n`)` denotes an address where $2^{n-1}/(2^{n-1}+1)$ of the remaining stack space since reference point `av` is exhausted (1/2 for $n = 1$, 2/3 for $n = 2$, etc.). The test `gc_needed(av,`n`)` becomes true whenever `avma` drops below that address.

4.3.7 Final comments.

PARI's manual garbage collection may seem cumbersome at first. But it is flexible and very fast, and compares favorably with methods used in other systems. Our benchmarks indicate that the price paid during garbage collections, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

It is of course harder on the programmer, and error-prone if you do not stick to a consistent PARI programming style. At first, just use `gc_GEN` or `gc_all` to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes. In any case, all garbage collections routines run in almost linear complexity in the size of the recovered data.

4.4 Creation of PARI objects, assignments, conversions.

4.4.1 Creation of PARI objects. The basic function which creates a PARI object is

`GEN cgetg(long l, long t)` l specifies the number of longwords to be allocated to the object, and t is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that "the PARI stack overflows" is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (for instance `affii` or `affrr`).

Note that the length argument l is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD`, and 5 for type `t_PADIC` and `t_QFB`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

Notes. 1) The main use of this function is create efficiently a constant object. Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

`GEN cgeti(long length)`

`GEN cgetr(long prec)`

should be used instead of `cgetg(length, t_INT)` and `cgetg(nbits2lg(prec), t_REAL)` respectively. Although `cgetg(length, t_REAL)` remains a possibility, if the `length` value is available, rather than `cgetr(bit_accuracy(length))`. Finally

`GEN cgetc(long prec)`

creates a `t_COMPLEX` whose real and imaginary part are `t_REAL`s allocated by `cgetr(prec)`.

Examples. 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose "precision" is `bit_accuracy(DEFAULTPREC)` $= 64$. This means `z` can hold rational integers of absolute value less than 2^{64}. Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy`(`MEDDEFAULTPREC`) $= 96$ bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for 4×3 matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function **cgetg** creates only the "root", and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

See also the technical functions `mkintn` and `mkvecsmalln` in the next chapters, which require great care with integer constants (whose type must be explicited).

Warning. Contrary to the policy of general PARI functions, the functions `mkvecn`, `mkcoln` and `mkpoln` are shallow: they do *not* copy their arguments. For instance

```
/* stack-clean: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);

/* not OK for gc_upto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);

/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
gunclone(x);
```

4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of `BITS_IN_LONG`-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless.

4.4.3 Assignment and copy. Firstly, if `x` and `y` are both declared as `GEN` (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure. The function which does this is called **gcopy**. Its syntax is:

```
GEN gcopy(GEN x)
```

and the effect is to create a new copy of x on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of `x` is enough, leaving pointers to the original data in `x` for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for `gc_upto` !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)` and `y = shallowcopy(x)`.

4.4.4 Clones. Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by garbage collection functions. The function which does this is called **gclone**. Its syntax is:

```
GEN gclone(GEN x)
```

A clone can be removed from the heap (thus destroyed) using

```
void gunclone(GEN x)
```

No PARI object should keep references to a clone which has been destroyed!

4.4.5 Conversions. The following functions convert C objects to PARI objects (creating them on the stack as usual):

`GEN stoi(long s)`: C `long` integer ("small") to `t_INT`.

`GEN dbltor(double s)`: C `double` to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of precision `DEFAULTPREC` = 64 bits, although only 53 of them are significant (the others are set to 0).

We also have the converse functions:

`long itos(GEN x)`: `x` must be of type `t_INT`,

`double rtodbl(GEN x)`: `x` must be of type `t_REAL`,

as well as the more general ones:

`long gtolong(GEN x)`,

`double gtodouble(GEN x)`.

4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI's handle on some mathematical entity, so we will shamelessly write $z \neq 0$ to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly not `NULL`). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* attached to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.4). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

`long typ(GEN z)` returns the type number of `z`.

`void settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

`long lg(GEN z)` returns the length (in longwords) of the root of `z`.

`long setlg(GEN z, long l)` sets the length of `z` to `l`; you should not have to use this function if you use `cgetg`.

`void lg_increase(GEN z)` increase the length of `z` by 1; you should not have to use this function if you use `cgetg`.

`long isclone(GEN z)` is `z` a clone?

`void setisclone(GEN z)` sets the *clone* bit.

`void unsetisclone(GEN z)` clears the *clone* bit.

Important remark. For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the "universal constants" (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

Remark. The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don't set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don't do that.

4.5.1 Type `t_INT` (integer). this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or -1 if `z` > 0, `z` $= 0$, `z` < 0 respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is "normalized", meaning that the most significant mantissa longword is nonzero. However, the integer may have been created with a longer length. Hence the "length" which is in `z[0]` can be larger than the "effective length" which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that `z` $\neq 0$, and let

$$|z| = \sum_{i=0}^{n} z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\texttt{BITS\_IN\_LONG}}.$$

With these notations, n is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`, z_n.

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`, z_0.

`GEN int_W(GEN z, long i)` returns the i-th significant word of `z`, z_i. Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the i-th significant word of `z`, z_i, assuming `lgefint(z)` is `lz` $(= n + 3)$. Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`, z_{i-1} assuming `z` points to z_i.

`GEN int_nextW(GEN z)` returns the next (more significant) word of `z`, z_{i+1} assuming `z` points to z_i.

Unnormalized integers, such that z_n is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

`void int_normalize(GEN z, long known0)`

normalizes in place a nonnegative integer (such that z_n is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
  long i, lx, ly, lout;
```

```
  long *xp, *yp, *outp; /* mantissa pointers */
  GEN out;

  if (!signe(x) || !signe(y)) return gen_0;
  lx = lgefint(x); xp = int_LSW(x);
  ly = lgefint(y); yp = int_LSW(y); lout = min(lx,ly); /* > 2 */

  out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
  outp = int_LSW(out);
  for (i=2; i < lout; i++)
  {
    *outp = (*xp) & (*yp);
    outp  = int_nextW(outp);
    xp    = int_nextW(xp);
    yp    = int_nextW(yp);
  }
  if ( !*int_MSW(out) ) out = int_normalize(out, 1);
  return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

4.5.2 Type `t_REAL` (real number). this type has a second codeword z[1] which also encodes its sign, obtained or set using the same functions as for a `t_INT`, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of `z`. This is defined even when `z` is equal to zero.

`void setexpo(GEN z, long e)` sets the exponent of `z` to `e`.

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for `z`, even if `z` is not a real number.

`long gsigne(GEN z)` which returns a sign for `z`, even when `z` is a real number of type `t_INT`, `t_FRAC` or `t_REAL`, an infinity (`t_INFINITY`) or a `t_QUAD` of positive discriminant.

The real zero is characterized by having its sign equal to 0. If `z` is not equal to 0, then it is represented as $\pm 2^e M$, where e is the exponent, and $M \in [1, 2[$ is the (normalized) significand of z, whose bits are stored in $\mathtt{z}[2], \ldots, \mathtt{z}[\mathtt{lg(z)} - 1]$.

More precisely, let m be the integer (`z[2]`,..., `z[lg(z)-1]`) in base `2^BITS_IN_LONG`; here, `z[2]` is the most significant longword and its most significant bit is 1. Then M is the rational number $m/2^{\mathtt{bit\_accuracy(lg(z))}-1-\mathtt{expo}(z)}$.

`long realprec(GEN x)` returns the precision of a `t_REAL` in bits, which is, which is the total number of bits which can *fit* in this `t_REAL`. This is 0 for a real 0 and a positive multiple of `BITS_IN_LONG` otherwise. There is potential for confusion here: many functions take a `prec` or `bitprec` argument to indicate the bit precision required in the output. If this requested bit precision is not a multiple of `BITS_IN_LONG`, then the number of correct bits in `t_REAL` components of the output will be *lower* than what `realprec` indicates. You will know the actual bit precision because it was given in the function call but there is no way to recover it exactly from a `t_REAL` output: `realprec` returns an upper bound (and the difference is less than `BITS_IN_LONG`).

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets `*e` to the exponent `realprec(z)` $- 1 -$ `expo`(z), so that $z = m/2^e$.

Thus, the real number 3.5 to accuracy `realprec(z)` is represented as `z[0]` (encoding `type` = `t_REAL`, `lg(z)`), `z[1]` (encoding `sign` = 1, `expo` = 1), `z`[2] = `0xe0000000`, `z`[3] = ... = `z`[`lg(z)` $- 1$] = `0x0`.

4.5.3 Type `t_INTMOD`. `z[1]` points to the modulus, and `z[2]` at the number representing the class `z`. Both are separate `GEN` objects, and both must be `t_INT`s, satisfying the inequality $0 \le$ `z`[2] $<$ `z`[1].

4.5.4 Type `t_FRAC` (rational number). `z[1]` points to the numerator n, and `z[2]` to the denominator d. Both must be of type `t_INT` such that $n \neq 0$, $d > 0$ and $(n, d) = 1$.

4.5.5 Type `t_FFELT` (finite field element). (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using `ffgen`.

The second codeword `z[1]` determines the storage format of the element, among

• `t_FF_FpXQ`: `A=z[2]` and `T=z[3]` are `FpX`, `p=z[4]` is a `t_INT`, where p is a prime number, T is irreducible modulo p, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_p[X]/T$.

• `t_FF_Flxq`: `A=z[2]` and `T=z[3]` are `Flx`, `l=z[4]` is a `t_INT`, where l is a prime number, T is irreducible modulo l, and $\deg A < \deg T$ This represents the element $A \pmod T$ in $\mathbf{F}_l[X]/T$.

• `t_FF_F2xq`: `A=z[2]` and `T=z[3]` are `F2x`, `l=z[4]` is the `t_INT` 2, T is irreducible modulo 2, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_2[X]/T$.

4.5.6 Type `t_COMPLEX` (complex number). `z[1]` points to the real part, and `z[2]` to the imaginary part. The components `z[1]` and `z[2]` must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for $p = 2$ or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

4.5.7 Type `t_PADIC` (p-adic numbers). this type has a second codeword `z[1]` which contains the following information: the p-adic precision (the exponent of p modulo which the p-adic unit corresponding to `z` is defined if `z` is not 0), i.e. one less than the number of significant p-adic digits, and the exponent of `z`. This information can be handled using the following functions:

`long precp(GEN z)` returns the p-adic precision of `z`. This is 0 if `z` $= 0$.

`void setprecp(GEN z, long d)` sets the p-adic precision of `z` to `d`.

`long valp(GEN z)` returns the p-adic valuation of `z` (i.e. the exponent). This is defined even if `z` is equal to 0.

`void setvalp(GEN z, long e)` sets the p-adic valuation of `z` to `e`.

The following accessors are available:

• `padic_p`(z): the prime p.

• `padic_pd`(z): p^d, where d is the p-adic precision

• `padic_u`(z): the p-unit u modulo p^d such that $z = p^v(u + O(p^d))$ or 0 (if and only if z is 0).

4.5.8 Type `t_QUAD` (quadratic number). `z[1]` points to the canonical polynomial P defining the quadratic field (as output by `quadpoly`), `z[2]` to the "real part" and `z[3]` to the "imaginary part". The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of `z` with respect to the canonical basis $(1, X)$ of $\mathbf{Q}[X]/(P(X))$. Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

4.5.9 Type `t_POLMOD` (polmod). as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of `z`. Both must be of type `t_POL` in the same variable, satisfying the inequality $\deg \mathtt{z}[2] < \deg \mathtt{z}[1]$. However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section 4.6) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable.

4.5.10 Type `t_POL` (polynomial). this type has a second codeword. It contains a "*sign*": 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for x, 1 for y, etc...).

These data can be handled with the following macros: **signe** and **setsigne** as for `t_INT` and `t_REAL`,

`long varn(GEN z)` returns the variable number of the object `z`,

`void setvarn(GEN z, long v)` sets the variable number of `z` to `v`.

The variable numbers encode the relative priorities of variables, we will give more details in Section 4.6. Note also the function `long gvar(GEN z)` which tries to return a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.

The components `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of the polynomial *in ascending order*, with `z[2]` being the constant term and so on.

For a `t_POL` of nonzero sign, `degpol`, `leading_coeff`, `constant_coeff`, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that this is a *shallow* construction: no copy is made on the PARI stack so the returned value is not safe for a basic `gc_upto` call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 10.7.

`long degree(GEN x)` returns the degree of `x` with respect to its main variable even when `x` is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is -1.

Important remark. The leading coefficient of a `t_POL` may be equal to zero:

• it is not allowed to be an exact rational 0, such as `gen_0`;

• an exact nonrational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;

• an inexact 0, like `0.E-38` or `O(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. This functions sets `signe` (to 0 or 1) properly.

Caveat. A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x)` < 0, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INT`s or `t_FRAC`s. The same remark applies to `t_SER`s.

4.5.11 Type `t_SER` (power series). This type also has a second codeword, which encodes a "*sign*", i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: **signe**, **setsigne**, **varn**, **setvarn** as for polynomials. Beware: do *not* use **expo** and **setexpo** on power series.

`long valser(GEN z)` returns the valuation of `z` This is defined even if `z` is equal to 0.

`void setvalser(GEN z, long e)` sets the valuation of `z` to `e`.

The coefficients `z[2]`, `z[3]`,... `z[lg(z)-1]` point to the coefficients of `z` in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.) A series whose coefficients are integers equal to zero is represented as $O(x^n)$ (`zeroser`(vx, n)). A series whose coefficients are exact zeroes, but not all of them integers (e.g. an `t_INTMOD` such as `Mod(0,2)`) is represented as $z * x^{n-1} + O(x^n)$, where z is the 0 of the base ring, as per `Rg_get_0`.

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

4.5.12 Type `t_RFRAC` (rational function). `z[1]` points to the numerator n, and `z[2]` on the denominator d. The denominator must be of type `t_POL`, with variable of priority at least as high as that of the numerator. The numerator n is not an exact 0 and $(n, d) = 1$ (see `gred_rfac2`).

4.5.13 Type `t_QFB` (binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form, and `z[4]` point to the form discriminant. All four are of type `t_INT`.

4.5.14 Type `t_VEC` and `t_COL` (vector). `z[1]`, `z[2]`,...`z[lg(z)-1]` point to the components of the vector.

4.5.15 Type `t_MAT` (matrix). `z[1]`, `z[2]`,...`z[lg(z)-1]` point to the column vectors of `z`, i.e. they must be of type `t_COL` and of the same length.

4.5.16 Type `t_VECSMALL` (vector of small integers). `z[1]`, `z[2]`,...`z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INT`s for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

4.5.17 Type `t_STR` (character string).

`char * GSTR(z)` (= `(z+1)`) points to the first character of the (`NULL`-terminated) string.

4.5.18 Type `t_ERROR` (error context). This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`,...`z[lg(z)-1]` are `GEN`s containing additional data, depending on the error type.

4.5.19 Type `t_CLOSURE` (closure). This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSURE`s are documented in Section 12.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

`long closure_is_variadic(GEN C)` returns 1 if the closure `C` is variadic, 0 else.

4.5.20 Type `t_INFINITY` (infinity).

This type has a single `t_INT` component, which is either 1 or -1, corresponding to $+\infty$ and $-\infty$ respectively.

`GEN mkmoo()` returns $-\infty$

`GEN mkoo()` returns ∞

`long inf_get_sign(GEN x)` returns 1 if x is $+\infty$, and -1 if x is $-\infty$.

4.5.21 Type `t_LIST` (list). this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

Implementation note. For the types including an exponent (or a valuation), we actually store a biased nonnegative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHEXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

4.6 PARI variables.

4.6.1 Multivariate objects.

We now consider variables and formal computations. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a "variable number". This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

```
int varncmp(long v, long w)
```

which is > 0, $= 0$, < 0 whenever v has lower, resp. same, resp. higher priority than w.

The way an object is considered in formal computations depends entirely on its "principal variable number" which is given by the function

```
long gvar(GEN z)
```

which returns a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the "outermost" number counts (we directly access `varn`(x) in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them* and do the right thing with the polynomials produced.

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it.

4.6.2 Creating variables. A basic difficulty is to "create" a variable. Some initializations are needed before you can use a given integer v as a variable number.

Initially, this is done for 0 and 1 (the variables `x` and `y` under `gp`), and $2, \ldots, 9$ (printed as `t2`, ... `t9`), with decreasing priority.

4.6.2.1 User variables. When the program starts, `x` (number 0) and `y` (number 1) are the only available variables, numbers 2 to 9 (decreasing priority) are reserved for building polynomials with predictable priorities.

To define further ones, you may use

```
GEN varhigher(const char *s)
```

```
GEN varlower(const char *s)
```

to recover a monomial of degree 1 in a new variable, which is guaranteed to have higer (resp. lower) priority than all existing ones at the time of the function call. The variable is printed as s, but is not part of GP's interpreter: it is not a symbol bound to a value.

* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, 'x' is defined in this way and has initially maximal priority (and variable number 0).

On the other hand

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `installed` or built-in function, or an alias. This function is mostly useless since it returns a variable with unpredictable priority. Don't use it to create new variables.

Caveat. You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

4.6.2.2 Temporary variables. You can create temporary variables using

`long fetch_var()` returns a new variable with *lower* priority than any variable currently in use.

`long fetch_var_higher()` returns a new variable with *higher* priority than any variable currently in use.

After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `t`number. You can assign a name to a temporary variable, after creating it, by calling the function

`void name_var(long n, char *s)`

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp` raises an error. Temporary variables are meant to be used as free variables to build polynomials and power series, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by garbage collection). When you do not need a variable number anymore, you can delete it using

`long delete_var()`

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if none remain). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```
long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);
```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use.

4.6.3 Comparing variables.

Let us go back to `varncmp`. There is an interesting corner case, when one of the compared variables (from `gvar`, say) is `NO_VARIABLE`. In this case, `varncmp` declares it has lower priority than any other variable; of course, comparing `NO_VARIABLE` with itself yields 0 (same priority);

In addition to `varncmp` we have

`long varnmax(long v, long w)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the highest priority. This function always returns a valid variable number unless it is comparing `NO_VARIABLE` to itself.

`long varnmin(long x, long y)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the lowest priority. Note that when comparing a true variable with `NO_VARIABLE`, this function returns `NO_VARIABLE`, which is not a valid variable number.

4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

4.7.1 Input.

For input, PARI provides several powerful high level functions which enable you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser.

There are two similar functions available to parse a string:

```
GEN gp_read_str(const char *s)
```

```
GEN gp_read_str_multiline(const char *s, char *last)
```

Both functions read the whole string `s`. The function `gp_read_str` ignores newlines: it assumes that the input is one expression and returns the result of this expression.

The function `gp_read_str_multiline` processes the text in the same way as the GP command `read`: newlines are significant and can be used to separate expressions. The return value is that of the last nonempty expression evaluated.

In `gp_read_str_multiline`, if `last` is not `NULL`, then `*last` receives the last character from the *filtered* input: this can be used to check if the last character was a semi-colon (to hide the output in interactive usage). If (and only if) the input contains no statements, then `*last` is set to `0`.

For both functions, gp's metacommands *are* recognized.

Two variants allow to specify a default precision while evaluating the string:

`GEN gp_read_str_prec(const char *s, long prec)` As `gp_read_str`, but set the bit precision to `prec` while evaluating *s*.

`GEN gp_read_str_bitprec(const char *s, long bitprec)` identical to `gp_read_str_prec`. DEPRECATED.

Note. The obsolete form

`GEN readseq(char *t)`

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a `GEN` from a file, you can use the simpler interface

`GEN gp_read_stream(FILE *file)`

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting `GEN`. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`. Return `NULL` when there are no more expressions to read (we reached EOF).

Finally, you can read in a whole file, as in GP's `read` statement

`GEN gp_read_file(char *name)`

As usual, the return value is that of the last nonempty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_flush()`: flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...)`: the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a `GEN`. For instance, the following are valid conversions for a `GEN` argument

`%Ps`	*convert to* `char*` *(will print an arbitrary* `GEN`*)*
`%P.10s`	*convert to* `char*`*, truncated to 10 chars*
`%P.2f`	*convert to floating point format with 2 decimals*
`%P4d`	*convert to integer, field width at least 4*

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a `GEN` which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its i-th `GEN` component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
pari_flush();
```

`void outmat(GEN x)` as above except if x is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of x are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `"%5Pd"` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `"%5.1Pf"` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENtostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENtoTeXstr(GEN x)`, suitable for inclusion in a TEX file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

```
void pari_vprintf(const char *fmt, va_list ap)
```

```
void pari_vfprintf(FILE *file, const char *fmt, va_list ap)
```

```
char* pari_vsprintf(const char *fmt, va_list ap)
```

4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exits the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

`void pari_err(numerr, ...)`

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if x is a `GEN` with internal type `t_STR`, say, `pari_err(e_TYPE,"extgcd", x)` prints the message:

```
***   incorrect type in extgcd (t_STR),
```

See Section 11.4 for details. In the libpari code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context attached to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and x (not only its type) as readily available components.

4.7.4 Warnings.

To issue a warning, use

`void pari_warn(warnerr, ...)` In that case, of course, we do *not* abort the computation, just print the requested message and go on. The basic example is

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC,...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem,  "bnfinit");
pari_warn(warnfile, "close", "afile");  /* error when closing "afile" */
```

4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree attached to an object:

`void dbgGEN(GEN x, long nb = -1)`, displays the recursive structure of `x`. If `nb` $= -1$, the full structure is printed, otherwise the leaves (nonrecursive components) are truncated to `nb` words.

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define oo
  call output((GEN)$arg0)
end
define xx
  call dbgGEN($arg0,-1)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the `GEN x` (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).

The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you `install` the latter under `gp`. Note that `DEBUGLEVEL` is redefined in each code module, attaching it to a particular debug domain (see `setdebug`).

`void setalldebug(long L)` sets all `DEBUGLEVEL` incarnations (all debug domains) to L.

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under gp.

4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect. Assume T was started by `timer_start`.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume T was started by `timer_start`.

`void walltimer_start(pari_timer *T)` start a timer, as if it had been started at the Unix epoch (see `getwalltime`).

`long walltimer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last checked. Assume T was started by `walltimer_start`.

`long walltimer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume T was started by `walltimer_start`.

`long timer_printf(pari_timer *T, char *format, ...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
  ...
  timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a nonreentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements `gp`'s timer and should not be used in libpari programs: `long gettime(void)` equivalent to `timer_delay`(T) attached to a private timer T.

4.8 Iterators, Numerical integration, Sums, Products.

4.8.1 Iterators. Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

- `fordiv` is a trivial iteration over a list produced by `divisors`.

- `forell`, `forqfvec` and `forsubgroup` are currently not implemented as an iterator but as a procedure with callbacks.

`void forell(void *E, long fun(void*, GEN), GEN a, GEN b, long flag)` goes through the same curves as `forell(ell,a,b,,flag)`, calling `fun(E, ell)` for each curve `ell`, stopping if `fun` returns a nonzero value.

`void forqfvec(void *E, long (*fun)(void *, GEN, GEN, double), GEN q, GEN b)` : Evaluate `fun(E,U,v,m)` on all v such that $q(Uv) < b$, where U is a `t_MAT`, v is a `t_VECSMALL` and $m = q(v)$ is a C double. The function `fun` must return 0, unless `forqfvec` should stop, in which case, it should return 1.

`void forqfvec1(void *E, long (*fun)(void *, GEN), GEN q, GEN b)`: Evaluate `fun(E,v)` on all v such that $q(v) < b$, where v is a `t_COL`. The function `fun` must return 0, unless `forqfvec` should stop, in which case, it should return 1.

`void forsubgroup(void *E, long fun(void*, GEN), GEN G, GEN B)` goes through the same subgroups as `forsubgroup(H = G, B,)`, calling `fun(E, H)` for each subgroup H, stopping if `fun` returns a nonzero value.

• `forprime` and `forprimestep`, iterators over primes and primes in arithmetic progressions, for which we refer you to the next subsection.

• `forcomposite`, we provide an iterator over composite integers:

`int forcomposite_init(forcomposite_t *T, GEN a, GEN b)` initialize an iterator T over composite integers in $[a, b]$; over composites $\geq a$ if $b =$ `NULL`. We must have $a \geq 0$. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise.

`GEN forcomposite_next(forcomposite_t *T)` returns the next composite in the range, assuming that T was initialized by `forcomposite_init`.

• `forvec`, for which we provide a convenient iterator. To initialize the analog of `forvec(X = v, ..., flag)`, call

`int forvec_init(forvec_t *T, GEN v, long flag)` initialize an iterator T over the vectors generated by `forvec(X =` v`,..., flag)`. This returns 0 if this vector list is empty, and 1 otherwise.

`GEN forvec_next(forvec_t *T)` returns the next element in the `forvec` sequence, or `NULL` if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector v are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than 10^9 tuples per minute. If speed is critical and all integers involved would fit in C `long`s, write a simple direct backtracking algorithm yourself.

• `forpart` is a variant of `forvec` which iterates over partitions. See the documentation of the `forpart` GP function for details. This function is available as a loop with callbacks:

`void forpart(void *data, long (*call)(void*, GEN), long k, GEN a, GEN n)`

It is also available as an iterator:

`void forpart_init(forpart_t *T, long k, GEN a, GEN n)` initializes an iterator over the partitions of k, with length restricted by n, and components restricted by a, either of which can be set to `NULL` to run without restriction.

`GEN forpart_next(forpart_t *T)` returns the next partition, or `NULL` when all partitions have been exhausted.

`GEN forpart_prev(forpart_t *T)` returns the previous partition, or `NULL` when all partitions have been exhausted.

In both cases, the partition must be used or copied before the next call since it is returned from a state array which will be modified in place. You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

• `forperm` to loop over permutations of k. See the documentation of the `forperm` GP function for details. This function is available as an iterator:

`void forperm_init(forperm_t *T, GEN k)` initializes an iterator over the permutations of k (`t_INT`, `t_VEC` or `t_VECSMALL`).

`GEN forperm_next(forperm_t *T)` returns the next permutation as a `t_VECSMALL` or `NULL` when all permutations have been exhausted. The permutation must be used or copied before the next call since it is returned from a state array which will be modified in place.

• `forsubset` to loop over subsets. See the documentation of the `forsubset` GP function for details. This function is available as two iterators:

`void forallsubset_init(forsubset_t *T, long n)`

`void forksubset_init(forsubset_t *T, long n, long k)`

It is also available in generic form:

`void forsubset_init(forsubset_t *T, GEN nk)` where `nk` is either a `t_INT` n or a `t_VEC` with two integral components $[n, k]$.

In all three cases, `GEN forsubset_next(forsubset_t *T)` returns the next subset as a `t_VECSMALL` or `NULL` when all subsets have been exhausted.

4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulong`s, the other as `GEN`. They are initialized as follows, where we expect to run over primes $\geq a$ and $\leq b$:

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where $b =$ `ULONG_MAX` means we will run through all primes representable in a `ulong` type.

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the `GEN` variant, where $b =$ `NULL` means $+\infty$.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator T over primes in an arithmetic progression, $p \geq a$ and $p \leq b$ (where $b =$ `NULL` means $+\infty$). The argument q is either a `t_INT` ($p \equiv a \pmod{q}$) or a `t_INTMOD Mod(c,N)` and we restrict to that congruence class.

All variants return 1 on success, and 0 if the iterator would run over an empty interval (if $a > b$, for instance). They allocate the `forprime_t` data structure on the PARI stack.

The successive primes are then obtained using

`GEN forprime_next(forprime_t *T)`, returns `NULL` if no more primes are available in the interval and the next suitable prime as a `t_INT` otherwise.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval and fitting in an `ulong` and the next suitable prime otherwise.

These two functions leave alone the PARI stack, and write their state information in the preallocated `forprime_t` struct. The typical usage is thus:

```
forprime_t T;
GEN p;
pari_sp av = avma, av2;

forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
```

```
while ( (p = forprime_next(&T)) )
{
  ...
  if ( prime_is_OK(p) ) break;
  set_avma(av2); /* delete garbage accumulated in this iteration */
}
set_avma(av); /* delete all */
```

Of course, the final `set_avma(av)` could be replaced by a `gc_upto` call. Beware that swapping the `av2 = avma` and `forprime_init` call would be incorrect: the first `set_avma(av2)` would delete the `forprime_t` structure!

4.8.3 Parallel iterators.

These iterators loop over the values of a `t_CLOSURE` taken at some data, where the evaluations are done in parallel.

• `parfor`. To initialize the analog of `parfor(i = a, b, ...)`, call

`void parfor_init(parfor_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of `code` on the integers between a and b.

`GEN parfor_next(parfor_t *T)` returns a `t_VEC [i,code(i)]` where i is one of the integers and `code(i)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parfor_next` must not be called anymore with the same initialization.

`void parfor_stop(parfor_t *T)` needs to be called when leaving the iterator before `parfor_next` returned `NULL`.

The following returns an integer $1 \le i \le N$ such that `fun(i)` is not zero, or `NULL`.

```
GEN
parfirst(GEN fun, GEN N)
{
  parfor_t T;
  GEN e;
  parfor_init(&T, gen_1, N, fun);
  while ((e = parfor_next(&T)))
  {
    GEN i = gel(e,1), funi = gel(e,2);
    if (!gequal0(funi))
    { /* found: stop the iterator and return the index */
      parfor_stop(&T);
      return i;
    }
  }
  return NULL; /* not found */
}
```

• `parforeach`. To initialize the analog of `parforeach(V, X, ...)`, call

`void parforeach_init(parforeach_t *T, GEN V, GEN code)` initialize an iterator over the evaluation of `code` on the components of V.

`GEN parforeach_next(parforeach_t *T)` returns a `t_VEC [V[i],code(V[i])]` where $V[i]$ is one of the components of V and `code(V[i])` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforeach_stop(parforeach_t *T)` needs to be called when leaving the iterator before `parforeach_next` returned `NULL`.

- `parforstep`. To initialize the analog of `parforstep(i = a, b, s, ...)`, call

`void parforstep_init(parforstep_t *T, GEN a, GEN b, GEN s, GEN code)`

initialize an iterator over the evaluation of `code` between a and b, with steps s (see `forstep` documentation for the possibilities for s).

`GEN parforstep_next(parforstep_t *T)`

returns a `t_VEC [x,code(x)]` where x is one of the evaluation points and `code(x)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforstep_next` must not be called anymore with the same initialization.

`void parforstep_stop(parforstep_t *T)` needs to be called when leaving the iterator before `parforstep_next` returned `NULL`.

- `parforprime`. To initialize the analog of `parforprime(p = a, b, ...)`, call

`void parforprime_init(parforprime_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of `code` on the prime numbers between a and b.

- `parforprimestep`. To initialize the analog of `parforprimestep(p = a, b, q, ...)`, call

`void parforprimestep_init(parforprime_t *T, GEN a, GEN b, GEN q, GEN code)` initialize an iterator over the evaluation of `code` on the prime numbers between a and b in the congruence class defined by q.

`GEN parforprime_next(parforprime_t *T)` returns a `t_VEC [p,code(p)]` where p is one of the prime numbers and `code(p)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforprime_stop(parforprime_t *T)` needs to be called when leaving the iterator before `parforprime_next` returned `NULL`.

- `parforvec`. To initialize the analog of `parforvec(X = V, ..., flag)`, call

`void parforvec_init(parforvec_t *T, GEN V, GEN code, long flag)` initialize an iterator over the evaluation of `code` on the vectors specified by `V` and `flag`, see `forvec` for detail.

`GEN parforvec_next(parforvec_t *T)` returns a `t_VEC [v,code(v)]` where v is one of the vectors and `code(v)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforvec_next` must not be called anymore with the same initialization.

`void parforvec_stop(parforvec_t *T)` needs to be called when leaving the iterator before `parforvec_next` returned `NULL`.

4.8.4 parallel data processing.

Below `f` and `g` are `t_CLOSURE`s of arity 1.

`GEN gen_parapply(GEN f, GEN V)` applies `f` to each components of V in parallel. Used to implement `parapply`.

`GEN gen_parapply_percent(GEN f, GEN V, long percent)` as `gen_parapply`, if `percent` not zero, print percentage of completion every `percent` steps. In `libpari`, `percent` should be 0 when `DEBUGLEVEL` is 0.

`GEN gen_parapply_slice(GEN g, GEN V, long m)` splits V in at most `m` chunks in arithmetic progressions, and calls `g` on each in parallel, and reassembles the result. `g` needs to apply `f` to each components of the chunk. The result is identical to `gen_parapply(f, V)` with a lower overhead. The usual choice of `m` is `mt_nbthreads()`.

`GEN gen_parapply_slice_zv(GEN g, GEN V, long m)` As `gen_parapply_slice`, but `g` must return `t_VECSMALL` and the output is a `t_VECSMALL` of `long` (`zv`).

`GEN gen_parpairwiseop(GEN h, GEN V)` `h` being a `t_CLOSURE` of arity 2, compute the vector $[h(V[1], V[2]), h(V[3], V[4]), \ldots]$ by apply `h` in parallel.

`GEN gen_parpairwiseop(GEN h, GEN V, long percent)` as `gen_parpairwiseop`, if `percent` not zero, print percentage of completion every `percent` steps. In `libpari`, `percent` should be 0 when `DEBUGLEVEL` is 0.

4.8.5 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```
void *E;
GEN (*eval)(void*, GEN)
```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions $f_t : x \to (x + t)^2$ via

```
GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }
```

One can then integrate f_1 between a and b with the call

```
intnum((void*)stoi(1), &fun, a, b, NULL, prec);
```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

4.9 Catching exceptions.

4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {
   recovery branch
}
pari_TRY {
   main branch
}
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 11.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

Restrictions.

• Such constructs can be nested without adverse effect, the innermost handler catching the exception.

• It is *valid* to leave either branch using `pari_err`.

• It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.

• In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception E returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error i-th component, as documented in Section 11.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */
   GEN x, E = pari_err_last();
   long code = err_get_num(E);
   if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
```

```
    x = err_get_compo(E, 2);
    /* e_INV has two components, 1: function name 2: noninvertible x */
    if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
    pari_CATCH_reset();
    return x; /* leave ! */
     ...
} pari_TRY {
    main branch
}
pari_ENDCATCH
```

4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **gcdext**.

Let x, y two integers and initially $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$, with $q = \texttt{floor}(x/y)$. Iterate until $y = 0$ in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \gcd(x, y).$$

In practice, there is no need to update s_y and t_y since $\gcd(x, y)$ and s_x are enough to recover s_y. The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper `Makefile`. You may ignore the first comment

```
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;
  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  return gc_all(av, 3, &a, U, V);
}
int
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```
  pari_sp av = avma;
  ...
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
    if (gc_needed(av,1))
      (void)gc_all(av, 4, &a, &b, &ux, &vx);
```

```
}
```

Chapter 5:
Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

Important advice. Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gc_upto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed. A `GEN` returned by a shallow function is called *not gc-safe*: one should use `gc_GEN` or `gc_all` to recover it when collecting garbage.

A function is *not stack-clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons. If is is nevertheless suitable for `gc_upto`, the function description will state it. Of course `gc_GEN` and `gc_all` are always able to handle it.

5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and 2^{16}. Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close pari, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `pari_init_primes` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`INIT_noINTGMPm`: do not install PARI-specific GMP memory functions. This option is ignored when the GMP library is not in use. You may install PARI-specific GMP memory functions later by calling

`void pari_kernel_init(void)`

and restore the previous values using

`void pari_kernel_close(void)`

This option should not be used without a thorough understanding of the problem you are trying to solve. The GMP memory functions are global variables used by the GMP library. If your program is linked with two libraries that require these variables to be set to different values, conflict ensues. To avoid a conflict, the proper solution is to record their values with `mp_get_memory_functions` and to call `mp_set_memory_functions` to restore the expected values each time the code switches from using one library to the other. Here is an example:

```
void *(*pari_alloc_ptr) (size_t);
void *(*pari_realloc_ptr) (void *, size_t, size_t);
void (*pari_free_ptr) (void *, size_t);
void *(*otherlib_alloc_ptr) (size_t);
void *(*otherlib_realloc_ptr) (void *, size_t, size_t);
void (*otherlib_free_ptr) (void *, size_t);

void init(void)
{
  pari_init(8000000, 500000);
  mp_get_memory_functions(&pari_alloc_ptr,&pari_realloc_ptr,
                          &pari_free_ptr);
  otherlib_init();
  mp_get_memory_functions(&otherlib_alloc_ptr,&otherlib_realloc_ptr,
                          &otherlib_free_ptr);
}
void function_that_use_pari(void)
{
```

```
  mp_set_memory_functions(pari_alloc_ptr,pari_realloc_ptr,
                          pari_free_ptr);
  /*use PARI functions*/
}
void function_that_use_otherlib(void)
{
  mp_set_memory_functions(otherlib_alloc_ptr,otherlib_realloc_ptr,
                          otherlib_free_ptr);
  /*use OTHERLIB functions*/
}
```

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT_SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT_DFTm`: frees the `GP_DATA` environment structure.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine. `INIT_noINTGMPm`: do not restore GMP memory functions.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_init_primes(ulong maxprime, ulong factorlimit)` Initialize the PARI primes, according to `maxprime` and `factorlimit`, either of which may be 0 (use a sensible default value). The parameters are updated then stored in `GP_DATA`: if is guaranteed on output that `maxprime` is less than or equal to `factorlimit` (it is silently increased to match it, if needed). See `initprimetable` for details.

This function is called by `pari_init(...,maxprime)`. It is provided for users calling `pari_init_opts` with the flag `INIT_noPRIMEm`.

`void pari_sighandler(int signum)` the actual signal handler that PARI uses. This can be used as argument to `pari_sig_init` or `signal(2)`.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```
int thread_start (...)
{
  long first_item_on_the_stack;
  ...
  pari_stackcheck_init(&first_item_on_the_stack);
}
```

`int pari_daemon(void)` forks a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

`void paristack_setsize(size_t rsize, size_t vsize)` sets the default `parisize` to `rsize` and the default `parisizemax` to `vsize`, and reallocate the stack to match these value, destroying its content. Generally used just after `pari_init`.

`void paristack_resize(ulong newsize)` changes the current stack size to `newsize` (double it if `newsize` is 0). The new size is clipped to be at least the current stack size and at most `parisizemax`. The stack content is not affected by this operation.

`void parivstack_reset(void)` resets the current stack to its default size `parisize`. This is used to recover memory after a computation that enlarged the stack. This function destroys the content of the enlarged stack (between the old and the new bottom of the stack). Before calling this function, you must ensure that `avma` lies within the new smaller stack.

`void paristack_newrsize(ulong newsize)` *(does not return)*. Library version of

```
default(parisize, "newsize")
```

Set the default `parisize` to `newsize`, or double `parisize` if `newsize` is equal to 0, then call `cb_pari_err_recover(-1)`.

`void parivstack_resize(ulong newsize)` *(does not return)*. Library version of

```
default(parisizemax, "newsize")
```

Set the default `parisizemax` to `newsize` and call `cb_pari_err_recover(-1)`.

5.1.3 Notions specific to the GP interpreter.

An `entree` is the generic object attached to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

`char *name`: the name under which the interpreter knows us.

`void *value`: a pointer to the C function to call.

`long menu`: a small integer ≥ 1 (to which group of function help do we belong, for the `?`n help menu).

`char *code`: the prototype code.

`char *help`: the help text for the function.

A routine in GP is described to the analyzer by an `entree` structure. Built-in PARI routines are grouped in *modules*, which are arrays of `entree` structs, the last of which satisfy `name = NULL` (sentinel). There are currently four modules in PARI/GP:

- general functions (`functions_basic`, known to `libpari`),
- gp-specific functions (`functions_gp`),

and two modules of obsolete functions. The function `pari_init` initializes the interpreter and declares all symbols in `functions_basic`. You may declare further functions on a case by case basis or as a whole module using

`void pari_add_function(entree *ep)` adds a single routine to the table of symbols in the interpreter. It assumes `pari_init` has been called.

`void pari_add_module(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter. It assumes `pari_init` has been called.

For instance, gp implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
```

A GP `default` is likewise attached to a helper routine, that is run when the value is consulted, or changed by `default0` or `setdefault`. Such routines are grouped in the module `functions_default`.

`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

5.1.4 Public callbacks.

The `gp` calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument s whenever PARI wants confirmation for action s, for instance in `secure` mode.

`long (*cb_pari_display_hist)(long n)` initialized to `NULL`. If set, called by `gp_embedded` in place of `gp_display_hist`.

`void (*cb_pari_init_histfile)(void)` initialized to `NULL`. Called when the `histfile` default is changed. The intent is for that callback to read the file content, append it to history in memory, then dump the expanded history to the new `histfile`.

`int (*cb_pari_is_interactive)(void)`; initialized to `NULL`.

`void (*cb_pari_quit)(long)` initialized to a no-op. Called when `gp` must evaluate the `quit` command.

`void (*cb_pari_start_output)(void)` initialized to `NULL`.

`int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument -1 on `SIGINT`, and argument `err` on error `err`. If it returns a nonzero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error. A possible simple-minded handler, used by the `gp` interpreter, is

`int gp_handle_exception(long err)` if the `breakloop` default is enabled (set to 1) and `cb_pari_break_loop` is not `NULL`, we call this routine with `err` argument and return the result.

`int (*cb_pari_err_handle)(GEN)` If not `NULL`, this routine is called with a `t_ERROR` argument from `pari_err`. If it returns a nonzero value, the error returns, in effect further ignoring the error, otherwise it raises a fatal error.

The default behavior is to print a descriptive error message (display the error), then return 0, thereby raising a fatal error. This differs from `cb_pari_handle_exception` in that the function is not called on `SIGINT` (which do not generate a `t_ERROR`), only from `pari_err`. Use `cb_pari_sigint` if you need to handle `SIGINT` as well.

The following function can be used by `cb_pari_err_handle` to display the error message.

`const char* closure_func_err()` return a statically allocated string holding the name of the function that triggered the error. Return NULL if the error was not caused by a function.

`int (*cb_pari_break_loop)(int)` initialized to `NULL`.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

```
pari_err(e_MISC, "user interrupt");
```

A possible simple-minded variant, used by the gp interpreter, is

`void gp_sigint_fun(void)`

`void (*cb_pari_pre_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument.

`void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument -1.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether s existed in older versions of `pari` (the `gp` callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- *flag* $= 0$: should print verbosely the answer, including help text if available.
- *flag* $= 1$: must return 0 if the function did not change, and a nonzero result otherwise. May print a help message.

`void (*cb_pari_long_help)(const char *s, long n)` It is called in place of the external help, to handle `??`.

5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the libpari library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

5.1.6 Utility functions.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

`char* gp_filter(const char *s)` pre-processor for the GP parser: filter out whitespace and GP comments from s. The returned string is allocated on the PARI stack and must not be freed.

`GEN pari_compile_str(const char *s)` low-level form of `compile_str`: assumes that s does not contain spaces or GP comments and returns the closure attached to the GP expression s. Note that GP metacommands are not recognized.

`int gp_meta(const char *s, int ismain)` low-level component of `gp_read_str`: assumes that s does not contain spaces or GP comments and try to interpret s as a GP metacommand (e.g. starting by `\` or `?`). If successful, execute the metacommand and return 1; otherwise return 0. The `ismain` parameter modifies the way `\r` commands are handled: if nonzero, act as if the file contents were entered via standard input (i.e. call `switchin` and divert `pari_infile`); otherwise, simply call `gp_read_file`.

`void pari_hit_return(void)` wait for the use to enter `\n` via standard input.

`void gp_load_gprc(void)` read and execute the user's `GPRC` file.

`void pari_center(const char *s)` print s, centered.

`void pari_print_version(void)` print verbose version information.

`long pari_community(void)` return the index of the support section n the help.

`const char* gp_format_time(long t)` format a delay of t ms suitable for `gp` output, with `timer` set. The string is allocated in the PARI stack via `stack_malloc`.

`const char* gp_format_prompt(const char *p)` format a prompt p suitable for `gp` prompting (includes colors and protecting ANSI escape sequences for readline).

`void pari_alarm(long s)` set an alarm after s seconds (raise an `e_ALARM` exception).

`void gp_help(const char *s, long flag)` print help for s, depending on the value of *flag*:

- `h_REGULAR`, basic help (`?`);
- `h_LONG`, extended help (`??`);
- `h_APROPOS`, a propos help (`??`).

`const char ** gphelp_keyword_list(void)` return a `NULL`-terminated array a strings, containing keywords known to `gphelp` besides GP functions (e.g. `modulus` or `operator`). Used by the online help system and the contextual completion engine.

`void gp_echo_and_log(const char *p, const char *s)` given a prompt p and attached input command s, update logfile and possibly print on standard output if `echo` is set and we are not in interactive mode. The callback `cb_pari_is_interactive` must be set to a sensible value.

`void gp_alarm_handler(int sig)` the `SIGALRM` handler set by the `gp` interpreter.

`void print_fun_list(char **list, long n)` print all elements of `list` in columns, pausing (hit return) every n lines. `list` is `NULL` terminated.

5.1.7 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

5.1.8 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t, long r)` adds `x` as the last history entry; t (resp. r) is the cpu (resp. real) time used to compute it.

`GEN pari_get_hist(long p)`, if $p > 0$ returns entry of index p (i.e. `%p`), else returns entry of index $n + p$ where n is the index of the last entry (used for `%`, `%‘`, `%‘‘`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the cpu time used to compute the history entry, instead of the entry itself.

`long pari_get_histrtime(long p)` as `pari_get_hist`, returning the real time used to compute the history entry, instead of the entry itself.

`GEN pari_histtime(long p)` return the vector `[cpu, real]` where `cpu` and `real` are as above.

`ulong pari_nb_hist(void)` return the index of the last entry.

`void gp_display_hist(long n)` print the history entry n using the usual GP format "$\%n = \ldots$".

`void str_display_hist(pari_str *S, long n)` as `gp_display_hist`, but output to the string S.

5.2 Handling GENs.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

5.2.1 Allocation.

`GEN cgetg(long l, long t)` allocates (the root of) a `GEN` of type t and length l. Sets $z[0]$.

`GEN cgeti(long l)` allocates a `t_INT` of length l (including the 2 codewords). Sets $z[0]$ only.

`GEN cgetr(long prec)` allocates a `t_REAL` of precision `prec`. Sets $z[0]$ only.

`GEN cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REAL`s of precision `prec`.

`GEN cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a `GEN` with the same type and length as x, setting `*lx` to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in $x[0]$ instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

`GEN vectrunc_init(long l)` perform `cgetg(l,t_VEC)`, then set the length to 1 and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vectrunc_append(GEN x, GEN y)` assuming x was allocated using `vectrunc_init`, appends y as the last element of x, which grows in the process. The function is shallow: we append y, not a copy; it is equivalent to

```
long lx = lg(x); gel(x,lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of x (the l argument to `vectrunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than $l - 1$ elements to x. Use the safer (but slower) `shallowconcat` when l is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(l, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct `GEN` until the vector is complete. Almost no PARI function will be able to operate on it.

`void vectrunc_append_batch(GEN x, GEN y)` successively apply

```
vectrunc_append(x, gel(y, i))
```

for all elements of the vector y.

`GEN coltrunc_init(long l)` as `vectrunc_init` but perform `cgetg(l,t_COL)`.

`GEN vecsmalltrunc_init(long l)`

`void vecsmalltrunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.

5.2.2 Length conversions.

These routines convert nonnegative lengths, often attached to floating point significands, between different units. Their behavior is undefined at negative integers. Strictly speaking the *length* of the `GEN` x is the output of `lg`(x) and counts a number of words, themselves comprising `BITS_IN_LONG` bits. By extension, a length could also be expressed in bits or even in decimal digits. We will not use the word "length" in this context but rather *number of bits* (`nbits`) or *bit precision* and *number of decimal digits*.

A `t_REAL` has a length, allowing to store a given number of bits, which is a multiple of `BITS_IN_LONG`, called *maximal bit precision*. This may be larger than the number of correct bits if a precision which is not a multiple of `BITS_IN_LONG` was requested in the function call producing it.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns $1 + $`floor`$(x \log_2 10/$`BITS_IN_LONG`$)$.

`long nbits2ndec(long x)` converts a number of bits to a number of decimal digits. Returns `floor`$(x \log_{10} 2)$.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing x bits, i.e., `ceil`$(x/$`BITS_IN_LONG`$)$.

`long nbits2lg(long x)` converts a number of bits to a `t_REAL` length in words. This is `nbits2nlong`$(x) + 2$ since a `t_REAL` has 2 code words.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing x bytes, i.e `ceil`$(x/$`sizeof(long)`$)$.

We can convert a `t_REAL` length to a bit precision without losing information. In the other direction, a conversion from bits to length (of a significand that can hold that many bits) rounds to the next multiple of `BITS_IN_LONG` and loses information: the original bit count can only be recovered with an error in $[0, $`BITS_IN_LONG`$ - 1[$. There is no error if the bit count is known to be a multiple of `BITS_IN_LONG`.

`long realprec(GEN x)` maximal bit precision of a `t_REAL`.

`long ndec2prec(long x)` converts a number of decimal digits to a maximal bit precision. This is `ndec2nlong`(x) times `BITS_IN_LONG`. Avoid using this function: it is used to approximately synchronize PARI's internal precision (`realbitprecision` in bits) and the decimal "pseudo-precision" (`realprecision`) provided as a convenience by `gp`'s user interface. Use bit precision throughout.

`long nbits2prec(long x)` convert x to the next multiple of `BITS_IN_LONG`. Avoid using this function, make sure your implementation works for an arbitrary bit precision, without rounding. An exception is when caching a `t_REAL` constant x for later use: computing it at a given maximal bit precision allows to check later whether the cache is sufficiently accurate by comparing the requested accuracy with `realprec`(x) (otherwise the cache accuracy would have to be maintained separately).

`long bit_accuracy(long x)` converts a `t_REAL` length into a number of significant bits. This is $(x - 2)$`BITS_IN_LONG`.

`double bit_accuracy_mul(long x, double y)` returns $(x - 2)(y \times$ `BITS_IN_LONG`$)$. This is equal to y`bit_accuracy`(x).

`long precdbl(long x)` return $2 * x$. Doubles a bit precision.

The following functions are *deprecated* and kept for backward compatibility. They will be removed in a future version:

`long bit_prec(GEN x)` alias for `realprec`.

`long lg2prec(long x)` alias for `bit_accuracy`.

`long nbits2extraprec(long x)` alias for `nbits2prec`.

`long prec2nbits(long x)` return x.

`double prec2nbits_mul(long x, double y)` returns $x \times y$.

`long prec2ndec(long x)` alias for `nbits2ndec`.

`long ndec2nbits(long x)` alias for `ndec2prec`.

`long prec2lg(long x)` alias for `nbits2lg`).

5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of `x`. The header files included through `pari.h` define symbolic constants for the `GEN` types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether `x` is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number `t` this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of `x` in `BITS_IN_LONG`-bit words.

`long lgefint(GEN x)` returns the effective length of the `t_INT` `x` in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign (-1, 0 or 1) of `x`. Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only 0 or 1 are possible).

`long gsigne(GEN x)` returns the sign of a real number x, valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC` and `t_QUAD` of positive discriminants. Raise a type error if `typ(x)` is not among those.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT` `x`. This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL` `x`.

`long mpexpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL` `x`.

`long gexpo(GEN x)` same as `expo`, but also valid when `x` is not a `t_REAL` (returns the largest exponent found among the components of `x`). When `x` is an exact 0, this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long gexpo_safe(GEN x)` same as `gexpo`, but returns a value strictly less than `-HIGHEXPOBIT` when the exponent is not defined (e.g. for a `t_PADIC` or `t_INTMOD` component).

`long valp(GEN x)` returns the p-adic valuation (for a `t_PADIC`).

`long valser(GEN x)` returns the X-adic valuation (for a `t_SER`).

`long precp(GEN x)` returns the precision of the `t_PADIC x`.

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER x` (between 0 and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object `x` (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which x is defined, e.g. if $x \in \mathbf{Z}[a][b]$ return (the variable number for) a. Return `NO_VARIABLE` if x has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL x` with respect to its main variable, *if* its leading coefficient is nonzero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely

- if `x` is not zero: the bit size of its mantissa;
- if `x` has sign 0: $\max(1, -expo(x))$.

If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision).

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns **gexpo(x)** multiplied by $\log_{10}(2)$. This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

5.2.4 Eval type-dependent information. These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GEN`s)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GEN`s). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in $-1, 0, 1$) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprecp(long x)` convert p-adic (X-adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprecp(long x)` same as `evalprecp` *without* the overflow check.

`long evalvalp(long x)` convert p-adic valuation `x` to bitmask (second codeword of `t_PADIC`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalvalser(long x)` convert X-adic valuation `x` to bitmask (second codeword of `t_SER`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalser(long x)` same as `evalvalser` *without* the overflow check.

`long evalexpo(long x)` convert exponent `x` to bitmask (second codeword of `t_REAL`). Raise overflow error if `x` is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length `x` to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

5.2.5 Set type-dependent information. Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of `x` to `s`.

`void setlg(GEN x, long s)` sets the length of `x` to `s`. This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT` `x` to `s`. The number `s` must be less than or equal to the length of `x`.

`void setsigne(GEN x, long s)` sets the sign of `x` to `s`. If `x` is a `t_INT` or `t_REAL`, `s` must be equal to -1, 0 or 1, and if `x` is a `t_POL` or `t_SER`, `s` must be equal to 0 or 1. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void togglesign(GEN x)` sets the sign s of `x` to $-s$, in place.

`void togglesign_safe(GEN *x)` sets the s sign of `*x` to $-s$, in place, unless `*x` is one of the integer universal constants in which case replace `*x` by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign s of `x` to $|s|$, in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of `*y` to that of `x`, in place, unless `*y` is one of the integer universal constants in which case replace `*y` by its negation if needed (e.g. replace `gen_1` by `gen_m1` if `x` is negative). No other sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void normalize_frac(GEN z)` assuming z is of the form `mkfrac(a,b)` with $b \neq 0$, make sure that $b > 0$ by changing the sign of a in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL` `x` to `s`. The value `s` must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the p-adic valuation of `x` to `s`, if `x` is a `t_PADIC`.

`void setvalser(GEN x, long s)` sets the X-adic valuation of `x` to `s`, if `x` is a `t_SER`, respectively.

`void setprecp(GEN x, long s)` sets the p-adic precision of the `t_PADIC` `x` to `s`.

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER` `x` to `s` (where $0 \leq$ `s` $\leq$ `MAXVARN`).

5.2.6 Type groups. In the following functions, `t` denotes the type of a `GEN`. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```
is_intreal_t(typ(x))
```

`int is_recursive_t(long t)` `true` iff `t` is a recursive type (the nonrecursive types are `t_INT`, `t_REAL`, `t_STR`, `t_VECSMALL`). Somewhat contrary to intuition, `t_LIST` is also nonrecursive, ; see the Developer's guide for details.

`int is_intreal_t(long t)` `true` iff `t` is `t_INT` or `t_REAL`.

`int is_rational_t(long t)` `true` iff `t` is `t_INT` or `t_FRAC`.

`int is_real_t(long t)` `true` iff `t` is `t_INT` or `t_REAL` or `t_FRAC`.

`int is_qfb_t(long t)` `true` iff `t` is `t_QFB`.

`int is_vec_t(long t)` `true` iff `t` is `t_VEC` or `t_COL`.

`int is_matvec_t(long t)` `true` iff `t` is `t_MAT`, `t_VEC` or `t_COL`.

`int is_scalar_t(long t)` `true` iff `t` is a scalar, i.e a `t_INT`, a `t_REAL`, a `t_INTMOD`, a `t_FRAC`, a `t_COMPLEX`, a `t_PADIC`, a `t_QUAD`, or a `t_POLMOD`.

`int is_extscalar_t(long t)` `true` iff `t` is a scalar (see `is_scalar_t`) or `t` is `t_POL`.

`int is_const_t(long t)` `true` iff `t` is a scalar which is not `t_POLMOD`.

`int is_noncalc_t(long t)` true if generic operations (`gadd`, `gmul`) do not make sense for t: corresponds to types `t_LIST`, `t_STR`, `t_VECSMALL`, `t_CLOSURE`

5.2.7 Accessors and components. The first two functions return `GEN` components as copies on the stack:

`GEN compo(GEN x, long n)` creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

`GEN truecoeff(GEN x, long n)` creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a `GEN` component. No copy is made on the stack:

`GEN constant_coeff(GEN x)` returns the address of the constant coefficient of `t_POL x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

`GEN leading_coeff(GEN x)` returns the address of the leading coefficient of `t_POL x`, i.e. the coefficient of largest index stored in the array representing x. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

`GEN gel(GEN x, long i)` returns the address of the `x[i]` entry of `x`. (`el` stands for element.)

`GEN gcoeff(GEN x, long i, long j)` returns the address of the `x[i,j]` entry of `t_MAT x`, i.e. the coefficient at row `i` and column `j`.

`GEN gmael(GEN x, long i, long j)` returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

`GEN gmael2(GEN A, long x1, long x2)` is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

The following routines return the component of `t_VECSMALL` as `ulong`.

`ulong uel(GEN x, long i)` returns the entry `x[i]` as a `ulong`, assuming x is a `t_VECSMALL`.

`ulong ucoeff(GEN x, long i, long j)` returns the `x[j][i]` entry of `x` as a `ulong` assuming `x[i][j]` is a `t_VECSMALL`.

`ulong umael(GEN x, long i, long j)` returns the `x[i][j]` entry of `x` as a `ulong` assuming `x[i]` is a `t_VECSMALL`.

`ulong umael2(GEN A, long x1, long x2)` is an alias for `umael`. Similar macros `umael3`, `umael4`, `umael5` are available.

5.3 Global numerical constants.

These are defined in the various public PARI headers.

5.3.1 Constants related to word size.

`long` BITS_IN_LONG $= 2^{\texttt{TWOPOTBITS\_IN\_LONG}}$: number of bits in a `long` (32 or 64).

`long` BITS_IN_HALFULONG: BITS_IN_LONG divided by 2.

`long` LONG_MAX: the largest positive `long`.

`ulong` ULONG_MAX: the largest `ulong`.

`long` DEFAULTPREC: the precision (`realprec`) of a `t_REAL` with 64 bits of accuracy

`long` MEDDEFAULTPREC: the precision (`realprec`) of a `t_REAL` with 128 bits of accuracy

`long` BIGDEFAULTPREC: the precision (`realprec`) of a `t_REAL` with 192 bits of accuracy

`ulong` HIGHBIT: the largest power of 2 fitting in an `ulong`.

`ulong` LOWMASK: bitmask yielding the least significant bits.

`ulong` HIGHMASK: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

`ulong LOWWORD(ulong a)` returns least significant bits.

`ulong HIGHWORD(ulong a)` returns most significant bits.

Finally

`long divsBIL(long n)` returns the Euclidean quotient of n by BITS_IN_LONG (with nonnegative remainder).

`long remsBIL(n)` returns the (nonnegative) Euclidean remainder of n by BITS_IN_LONG

`long dvmdsBIL(long n, long *r)`

`ulong dvmduBIL(ulong n, ulong *r)` sets r to `remsBIL(n)` and returns `divsBIL(n)`.

5.3.2 Masks used to implement the `GEN` type.

These constants are used by higher level macros, like `typ` or `lg`:

EXPOnumBITS, LGnumBITS, SIGNnumBITS, TYPnumBITS, VALPnumBITS, VARNnumBITS: number of bits used to encode `expo`, `lg`, `signe`, `typ`, `valp`, `varn`.

PRECPSHIFT, SIGNSHIFT, TYPSHIFT, VARNSHIFT: shifts used to recover or encode `precp`, `varn`, `typ`, `signe`

CLONEBIT, EXPOBITS, LGBITS, PRECPBITS, SIGNBITS, TYPBITS, VALPBITS, VARNBITS: bitmasks used to extract `isclone`, `expo`, `lg`, `precp`, `signe`, `typ`, `valp`, `varn` from `GEN` codewords.

MAXVARN: the largest possible variable number.

NO_VARIABLE: sentinel returned by `gvar(x)` when `x` does not contain any polynomial; has a lower priority than any valid variable number.

HIGHEXPOBIT: a power of 2, one more that the largest possible exponent for a `t_REAL`.

HIGHVALPBIT: a power of 2, one more that the largest possible valuation for a `t_PADIC` or a `t_SER`.

5.3.3 $\log 2$, π.

These are `double` approximations to useful constants:

`M_PI`: π.

`M_LN2`: $\log 2$.

`LOG10_2`: $\log 2/\log 10$.

`LOG2_10`: $\log 10/\log 2$.

5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a prime table `pari_PRIMES` is initialized with the successive primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly $4\texttt{maxprime}/\log(\texttt{maxprime})$ bytes in memory, so be sensible when choosing `maxprime`; it is 2^{20} by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that $\texttt{maxprime} < 2^{\texttt{BITS\_IN\_LONG}} - 2048$, whatever memory is available. In fact, `maxprime` is automatically replaced by

$$\max(\texttt{maxprime}, 65557) \ .$$

In particular, PARI guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero in the `pari_init` call.

Some convenience functions:

`ulong maxprime(void)` the largest prime computable using our prime table.

`ulong maxprimeN(void)` the index N of the largest prime computable using the prime table. I.e., $p_N = \texttt{maxprime}()$.

`GEN prodprimes(void)` a vector T where `T[i]` is the product of the primes in our prime table that are smaller than 2^{7+i}. Used to quickly detect small prime divisors of large integers, using gcd's.

`GEN prodprimeslim(void)` a vector L where `L[i]` is the largest prime less that 2^{7+i}.

`ulong maxprimelim(void)` the argument used for the last `initprimetable` call. This is at least 65537 and the largest prime less than or equal to this number is `maxprime`().

`long PRIMES_search(ulong x)` assumes $x \leq \texttt{maxprimelim}()$. Return $i > 0$ such that $x = \texttt{pari\_PRIMES}[i]$ iff x is prime. Else return $-i < 0$ such that x lies (strictly) between the $(i-1)$-th and the i-th prime; this is understood as $i = 1$ when $x < 2$: the statement about the non-existing 0-th prime is disregarded.

`void maxprime_check(ulong B)` raise an error if `maxprime()` is $< B$.

`void initprimetable(ulong maxp, ulong factorlimit)` computes the prime table `pari_PRIMES` (of all primes $p <$ `maxp`); time complexity is softly linear in `maxp`. We don't recommend values greater than 2^{32}: prime iterators produce fast primes up to $\texttt{maxp}^2$.

This function also caches data allowing fast trial division up to `factorlimit`; time complexity is also softly linear in `factorlimit` with much larger implied constants. Value greater than 2^{32}

will not generally be useful: for such values, expect to increase startup time by 15s and memory use by 10GB.

We recommend to set `factorlimit` $\leq$ `maxp`; otherwise, the parameter `factorlimit` is silently decreased to match `maxp`. A generic value of 2^{20} is quite sensible for `factorlimit`, for a negligible cost at startup (less than 50ms). Only increase this if your own code will impose *many* trial divisions up to the given limit: PARI itself uses other algorithms to detect factors larger than 2^{20}.

5.5 Handling the PARI stack.

5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length `n` and type `t`, and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetr(long b)` allocates memory on the stack for a `t_REAL` with `b` bits of accuracy, and initializes its first codeword. Identical to `cgetg(nbits2lg(b),t_REAL)`.

`GEN cgetc(long b)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s allocated with `cgetr(b)`.

`GEN new_chunk(size_t n)` allocates a `GEN` with n components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address `((GEN)avma) - n`, after checking that it is larger than `(GEN)bot`.

`void new_chunk_resize(size_t x)` this function is called by `new_chunk` when the PARI stack overflows. There is no need to call it manually. It will either extend the stack or report an `e_STACK` error.

`char* stack_malloc(size_t n)` allocates memory on the stack for n chars (*not* n `GEN`s). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `set_avma(oldavma)` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack. The memory is aligned on `sizeof(long)` bytes boundaries.

`char* stack_malloc_align(size_t n, long k)` as `stack_malloc`, but the memory is aligned on `k` bytes boundaries. The number`k` must be a multiple of the `sizeof(long)`.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

`char* stack_calloc_align(size_t n, long k)` as `stack_malloc_align`, setting the memory to zero.

Objects allocated through these last three functions cannot be handled by garbage collection routines, since they are not yet valid `GEN`s: their codewords must be filled first.

`GEN cgetalloc(size_t l, long t)`, same as `cgetg(l, t)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting `GEN` is now impervious to garbage collecting routines, but should be freed using `pari_free`.

5.5.2 Stack-independent binary objects.

`GENbin* copy_bin(GEN x)` copies x into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

`GENbin* copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

`GEN bin_copy(GENbin *p)` assuming p was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores x on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

`GEN GENbinbase(GENbin *p)` the `GEN` data actually stored in p. All addresses are stored as offsets with respect to a common reference point, so the resulting `GEN` is unusable unless it is a nonrecursive type; private low-level routines must be called first to restore absolute addresses.

`void shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

`void shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The `GENbin` type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

`GEN copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in C. If C = `NULL` return a linking table corresponding to the content of all gp variables. C may then be dumped to disk in binary form, for instance.

`void bincopy_relink(GEN C, GEN V)` given a binary object C, as dumped by writebin and read back into a session, and a linking table V, restore all closures contained in C (function pointers are translated to their current value).

5.5.3 Garbage collection.

See Section 4.3 for a detailed explanation and many examples.

`void set_avma(ulong av)` reset `avma` to `av`. You may think of this as a simple `avma = av` statement, but PARI developpers modify this statement in special code branches to detect garbage collecting issues (by invalidating the PARI stack below `av`).

`ulong get_avma(void)` return `avma`. Useful for languages that do not provide access to TLS variables.

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp av, pari_sp av2, GEN x)` obsolete low level garbage collector. Don't use this. In proper code, this can be replaced by `gc_upto(av, x)`

`GEN gc_GEN(pari_sp av, GEN x)` cleans up the stack from `av` on, recopying the object `x` to a new address which is returned.

`GEN gc_INT(pari_sp av, GEN x)` faster variant of `gc_GEN` when `x` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

`GEN gc_INT_affii(pari_sp av, GEN x, GEN *z)` This routine is technical, meant to reduce stack use when we modify big structures in place, for instance in linear algebra over the integers. Let `t_INT`s `x` and `*z` be such that `*z` was created *before* `av` was recorded. Then this function sets $*z \leftarrow x$ and collects accumulated garbage. This is done as follows:

• if `*z` is on the stack and large enough to store `x`, do so using `affii`. This destroys the *content* of `z` and leave a clean stack with `avma` = `av`.

• else set `*z = gc_INT(av, x)`. Compared to the previous case, we cleaned up the stack from `av` on, but the old `*z` is still taking up space that couldn't be (easily) reclaimed.

In a nutshell, we perform a `gc_INT` while taking advantage of the opportunity of overwriting `*z` without growing the stack, if possible.

`GEN gc_NULL(pari_sp av)` reset `avma` to `av` and return `NULL`.

The following 6 functions reset `avma` to `av` and return x:

`int gc_bool(pari_sp av, int x)`

`double gc_double(pari_sp av, double x)`

`int gc_int(pari_sp av, int x)`

`long gc_long(pari_sp av, long x)`

`ulong gc_ulong(pari_sp av, ulong x)` This allows for instance to return `gc_ulong(av, itou(z))`, whereas

```
pari_sp av = avma;
GEN z = ...
set_avma(av);
return itou(z);
```

should be frowned upon since `set_avma(av)` conceptually destroys everything from the reference point on, including z.

`GEN gc_const(pari_sp av, GEN x)` assumes that x is either not on the stack (clone, universal constant such as `gen_0`) or was defined before `av`.

`GEN gc_stoi(pari_sp av, long x)` reset `avma` to `av` and return `stoi(x)`.

`GEN gc_utoi(pari_sp av, long x)` reset `avma` to `av` and return `utoi(x)`.

`GEN gc_utoipos(pari_sp av, long x)` reset `avma` to `av` and return `utoipos(x)`.

`void retgc_const(pari_sp av, GEN x)` (macro) reset the stack to av before evaluating x and returning it.

`GEN gc_all(pari_sp av, int n, ...)`. Assumes that $1 \leq n$. This functions expects n further `GEN*` arguments: the stack is cleaned and the corresponding `GEN` are copied to the stack starting from `av` (in this order: the first argument comes first), and the first such `GEN` is returned. To be used in the following scenario:

```
GEN f(..., GEN *py)
{
  pari_sp av = avma;
  GEN x = ..., y = ...
  *py = y; return gc_all(av, 2, &x, py);
```

```
}
```

This function returns x, and the user also recovers y as a side effect. Not that we can later use `cgiv`(y) to recover the memory used by y while still keeping x.

`GEN gc_all_unsafe(pari_sp av, pari_sp ltop, int n, ...)` unsafe variant of `gc_all`, about twice faster. This is subject to the same restrictions as `gc_upto`, the only difference being that more than one address gets updated. Cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. To be used in the following scenario:

```
GEN f(..., GEN *py)
{
  ltop = avma;
  ... junk ...
  av = avma;
  x = fun1();  // gc clean
  y = fun2();  // gc clean
  *py = y; return gc_all_unsafe(av, ltop, 2, &x, py);
}
```

This function returns x, and we also recover y as a side effect. Not that we can later use `cgiv`(y) to recover the memory used by y while still keeping x.

`void gc_slice(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0]`, ..., `x[n-1]` (which are `GEN`s).

`void gc_slice_unsafe(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0]`, ..., `x[n-1]` (which are `GEN`s). Same assumptions as in `gc_all_unsafe`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
gel(z,1) = fun1();
gel(z,2) = fun2();
gc_slice_unsafe(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage();
z = cgetg(3, t_COMPLEX);
gel(z,1) = fun1();
gel(z,2) = fun2(); return gc_upto(av, z);
```

`GEN gc_upto(pari_sp av, GEN q)` analogous to (but faster than) `gc_GEN`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `set_avma(av)`; in particular, sentinels which are not even proper `GEN`s such as `q = NULL` are allowed.

`GEN gc_leaf(pari_sp av, GEN q)` assumes only that `q` is a leaf, i.e a nonrecursive type (`is_recursive_t(typ(q))` is nonzero). Contrary to `gc_INT` and `gc_upto`, `gc_leaf` leaves length and effective length of a `t_INT` unchanged.

5.5.4 Garbage collection: advanced use.

`void stackdummy(pari_sp av, pari_sp ltop)` inhibits the memory area between `av` *included* and `ltop` *excluded* with respect to garbage collection. This avoids a garbage collection which would copy objects around, but the stack space between `av` and `ltop` is not reclaimed.

More precisely, this routine assumes that `av` is recorded earlier than `ltop`, then marks the specified stack segment as a nonrecursive type of the correct length. Thus garbage collection functions will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gc_upto(av, gel(z,2));
```

or even more wasteful

```
z = gc_GEN(av0, z);
```

we temporarily lose (`av` − `ltop`) words but save a costly `gc_upto`. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the `stackdummy`, if the [`av`, `ltop`] zone is arbitrary (not even valid `GEN`s as could happen after direct truncation via `setlg`), we would leave dangerous data in the middle of `z`, which would be a problem for a later

```
gc_upto(..., z);
```

And even if it were made of valid `GEN`s, inhibiting the area makes sure `gc_stack_update` will not inspect their components, saving time.

Another natural use in low-level routines is to "shorten" an existing `GEN z` to its first `n` − 1 components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last `n` components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing `GEN`) is in fact so common, that we provide a function for this:

`void fixlg(GEN z, long ly)` a safe variant of `setlg(z, ly)`. If `ly` is larger than `lg(z)` do nothing. Otherwise, shorten z in place, using `stackdummy` to avoid later garbage collection problems.

`GEN gcopy_avma(GEN x, pari_sp *AVMA)` return a copy of x as from `gcopy`, except that we pretend that initially `avma` is `*AVMA`, and that `*AVMA` is updated accordingly (so that the total size of x is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a

pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

5.5.5 Debugging the PARI stack.

`int chk_gc_upto(GEN x)` returns 1 if `x` is suitable for `gc_upto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_stack(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gc_GEN_unsafe(...,ltop,...)`.

`void dbg_gc_upto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gc_upto(...,q)`.

`void dbg_fill_stack(void)` marks the unused portion of the stack (between its bottom and `avma`) with repeated magic values: `0xBADC0FFEE0DDF00D` on 64-bit archs, and `0xDEADBEEF` on 32-bit. This allows to quickly detect garbage collection errors, e.g., objets one of whose component would be overwritten later or uninitialized memory access. The `valgrind` framework offers more thorough possibilities but requires instrumenting the code.

5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of x on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of x on the stack, pretending that `lg(x)` is l, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` `true` iff x belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets `y = gcopy(x)` if x belongs to the stack, and `y = x` otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming `x` is a `t_INT`.

5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

5.6 The PARI heap.

5.6.1 Introduction.

It is implemented as `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate x and return $x +$ `headersize`). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set. The collection of blocks is maintained as a doubly-linked list (in creation order) *and* as an AVL tree (allowing to check in logarithmic time whether a given object / address is a block).

5.6.2 Public interface.

`GEN newblock(size_t n)` allocates a block of n *words* (not bytes).

`void killblock(GEN x)` deletes the block x created by `newblock`. Fatal error if x not a block.

`GEN gclone(GEN x)` creates a new permanent copy of x on the heap (allocated using `newblock`). The *clone bit* of the result is set.

`GEN gcloneref(GEN x)` if x is not a clone, clone it and return the result; otherwise, increase the clone reference count and return x.

`void gunclone(GEN x)` deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for `killblock`, but it is cleaner to kill clones (valid `GEN`s) using this function, and other blocks using `killblock`.

`void guncloneNULL(GEN x)` same as `gunclone`, first checking whether x is `NULL` (and doing nothing in this case).

`void gunclone_deep(GEN x)` is only useful in the context of the GP interpreter which may replace arbitrary components of container types (`t_VEC`, `t_COL`, `t_MAT`, `t_LIST`) by clones. If x is such a container, the function recursively deletes all clones among the components of x, then unclones x. Useless in library mode: simply use `gunclone`.

`void guncloneNULL_deep(GEN x)` same as `gunclone_deep`, first checking whether x is `NULL` (and doing nothing in this case).

`void traverseheap(void(*f)(GEN, void *), void *data)` this applies `f(`x`, data)` to each object x on the PARI heap, most recent first. Mostly for debugging purposes.

`GEN getheap()` a simple wrapper around `traverseheap`. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

`GEN cgetg_block(long x, long y)` as `cgetg(x,y)`, creating the return value as a `block`, not on the PARI stack.

`GEN cgetr_block(long prec)` as `cgetr(prec)`, creating the return value as a `block`, not on the PARI stack.

5.6.3 Implementation note. The hidden block header is manipulated using the following private functions:

`void* bl_base(GEN x)` returns the pointer that was actually allocated by `malloc` (can be freed).

`long bl_refc(GEN x)` the reference count of x: the number of pointers to this block. Decremented in `killblock`, incremented by the private function `void gclone_refc(GEN x)`; block is freed when the reference count reaches 0.

`long bl_num(GEN x)` the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until $2^{\texttt{BITS\_IN\_LONG}}$ blocks have been allocated and this wraps around.

`GEN bl_next(GEN x)` the block *after* x in the linked list of blocks (`NULL` if x is the last block allocated not yet killed).

`GEN bl_prev(GEN x)` the block allocated *before* x (never `NULL`).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write `bl_refc(x)++` for instance.

5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: 0–`nvar` (user variables), `max_avail`–`MAXVARN` (temporary variables), and `nvar+1`–`max_avail-1` (pool of free variable numbers).

5.7.1 Low-level.

`void pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes `x`. Don't use it.

`void pari_var_close(void)` attached destructor, called by `pari_close`.

`long pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

`long pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

`long pari_var_create(entree *ep)` low-level initialization of an `EpVAR`. Return the attached (new) variable number.

`GEN vars_sort_inplace(GEN z)` given a `t_VECSMALL` z of variable numbers, sort z in place according to variable priorities (highest priority comes first).

`GEN vars_to_RgXV(GEN h)` given a `t_VECSMALL` z of variable numbers, return the `t_VEC` of `pol_x`$(z[i])$.

5.7.2 User variables.

`long fetch_user_var(char *s)` returns a user variable whose name is `s`, creating it is needed (and using an existing variable otherwise). Returns its variable number.

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Return `NULL` for a temporary variable.

`entree* is_entry(const char *s)` returns the `entree*` attached to an identifier `s` (variable or function), from the interpreter hashtables. Return `NULL` is the identifier is unknown.

5.7.3 Temporary variables.

`long fetch_var(void)` returns the number of a new temporary variable (decreasing `max_avail`).

`long delete_var(void)` delete latest temp variable created and return the number of previous one.

`void name_var(long n, char *s)` rename temporary variable number `n` to `s`; mostly useful for nicer printout. Error when trying to rename a user variable.

5.8 Adding functions to PARI.

5.8.1 Nota Bene. As mentioned in the `COPYING` file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the `Configure` script and anything in the `config/` subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from `gp`.

5.8.2 Coding guidelines. Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a `GEN` and you want it to be accessible to `gp`, you have to make sure this `GEN` is suitable for `gc_upto` (see Section 4.3).

If error messages or warnings are to be generated in your function, use `pari_err` and `pari_warn` respectively. Recall that `pari_err` does not return but ends with a `longjmp` statement. As well, instead of explicit `printf` / `fprintf` statements, use the following encapsulated variants:

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_printf(const char *fmt, ...)`: write following arguments to the output stream, according to the conversion specifications in format `fmt` (see `printf`).

`void err_printf(const char *fmt, ...)`: as `pari_printf`, writing to PARI's current error stream.

`void err_flush(void)` flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from C. The other functions should be declared `static` in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the `Makefile` coming with the `extgcd` example in the distribution). It is however still inaccessible from `gp`.

5.8.3 GP prototypes. A *GP prototype* is a character string describing all the GP bytecode compiler needs to know about the function prototype. It contains a sequence of the following atoms:

• Return type: `GEN` by default (must be valid for `gc_upto`), otherwise the following can appear as the *first* char of the code string:

- `i` return `int`
- `l` return `long`
- `u` return `ulong`
- `v` return `void`
- `m` return a `GEN` which is not gc-safe.

The `m` code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

• Mandatory arguments, appearing in the same order as the input arguments they describe:

- `G` `GEN`
- `&` `*GEN`
- `L` `long` (we implicitly typecast `int` to `long`)
- `U` `ulong`
- `V` loop variable
- `n` variable, expects a variable number (a `long`, not an `*entree`)
- `W` a `GEN` which is a lvalue to be modified in place (for `t_LIST`)
- `r` raw input (treated as a string without quotes). Quoted args are copied as strings
  Stops at first unquoted `')'` or `','`. Special chars can be quoted using `'\'`
  Example: `aa"b\n)"c` yields the string `"aab\n)c"`
- `s` expanded string. Example: `Pi"x"2` yields `"3.142x2"`
  Unquoted components can be of any PARI type, converted to string following current output format
- `I` inline closure whose value is ignored, as in `for` loops,
  to be processed by `void closure_evalvoid(GEN C)`
- `E` inline closure whose value is used, as in `sum` loops,
  to be processed by `void closure_evalgen(GEN C)`
- `J` closure of arity 1, as in `parsum` loops,
  to be processed by `void closure_callgen1(GEN C)`

Closures are `GEN` of type `t_CLOSURE`.

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the `closure_call`*xxx* functions. A C function can be converted to a closure by using `strtoclosure`.

An *inline closure* is a GP expression in compiled (bytecode) form. It can be efficiently evaluated using the `closure_eval`*xxx* functions. A C function can be converted to a closure by using `strtoclosure`. and then to a pseudo-inline closure suitable for `I` and `E` by using `closuretoinl`.

In both cases, the arity of the closure needs to match the number of `V` atoms appearing before the code `I`, `E` or `J` in the prototype.

• Automatic arguments:

- `f` Fake `*long`. C function requires a pointer but we do not use the resulting `long`
- `p` current real precision in bits
- `P` series precision (default `seriesprecision`, global variable `precdl` for the library)
- `C` lexical context (internal, for `eval`, see `localvars_read_str`)

• Syntax requirements, used by functions like `for`, `sum`, etc.:
  `=` separator `=` required at this point (between two arguments)

• Optional arguments and default values:
  `E*` any number of expressions, possibly 0 (see `E`)
  `s*` any number of strings, possibly 0 (see `s`)
  `D`*xxx* argument can be omitted and has a default value

The `E*` code reads all remaining arguments in closure context and passes them as a single `t_VEC`. The `s*` code reads all remaining arguments in *string context* and passes the list of strings as a single `t_VEC`. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: `"xx" 1, "yy"`, the `s*` atom sends `[a, b, c]`, where a, b, c are `GEN`s of type `t_STR` (content `"xx"`), `t_INT` (equal to 1) and `t_STR` (content `"yy"`).

The format to indicate a default value (atom starts with a `D`) is "`D`*value*`,`*type*`,`", where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance `D0,L,` stands for "this optional argument is converted to a `long`, and is `0` by default". So if the user-given argument reads `1 + 3` at this point, `4L` is sent to the function; and `0L` if the argument is omitted. The following special notations are available:

`DG` optional `GEN`, send `NULL` if argument omitted.
`D&` optional `*GEN`, send `NULL` if argument omitted. The argument must be prefixed by `&`.
`DI`, `DE` optional closure, send `NULL` if argument omitted.
`DP` optional `long`, send `precdl` if argument omitted.
`DV` optional loop variable.
`Dn` optional variable number, -1 if omitted.
`Dr` optional raw string, send `NULL` if argument omitted.
`Ds` optional `char *`, send `NULL` if argument omitted.

Hardcoded limit. C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under `gp`, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (`gp` outputs an error message it the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

```
GEN f(GEN x, GEN y, long prec)   ----> "GGp"
void f(GEN x, GEN y, long prec)  ----> "vGGp"
void f(GEN x, long y, long prec) ----> "vGLp"
long f(GEN x)                    ----> "lG"
int f(long x)                    ----> "iL"
```

If you want more examples, `gp` gives you easy access to the prototype attached to all GP functions: just type `\h` *function*. You can then compare with the C prototypes as they stand in `paridecl.h`.

Remark. If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the bytecode evaluator implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

5.8.4 Integration with `gp` as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines is Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gc_upto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (_), the first character being alphabetic.

Then figure out the correct prototype code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

The addhelp part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (or `gprc`), and that's it. You can now use the new function *gpname* under `gp`, and we would very much like to hear about it!

Example. A complete description could look like this:

```
{
  install(bnfinit0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfinit0` . The function has one mandatory argument, and possibly two more (two `'D'` in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (`0` is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfinit0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

Important note. You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid `GEN` pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

• Fetch the complete sources of the PARI distribution.

• Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.

• Choose a help section and add a file `src/functions/`*section*`/`*gpname* containing the following, keeping the notation above:

`Function:` *gpname*
`Section:` *section*
`C-Name:` *libname*
`Prototype:` *code*
`Help:` *some help text*

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

• Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

Example. We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

`functions/number_fields/ClassGroupInit`

containing the following:

```
Function: ClassGroupInit
Section: number_fields
C-Name: bnfinit0
Prototype: GD0,L,DGp
Help: ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.

5.9 Globals related to PARI configuration.

5.9.1 PARI version numbers.

`paricfg_version_code` encodes in a single `long`, the Major and minor version numbers as well as the patchlevel.

`long PARI_VERSION(long M, long m, long p)` produces the version code attached to release $M.m.p$. Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

`PARI_VERSION_SHIFT` is the number of bits used to store each of the integers M, m, p in the version code.

`paricfg_vcsversion` is a version string related to the revision control system used to handle your sources, if any. For instance `git-`*commit hash* if compiled from a git repository.

The two character strings `paricfg_version` and `paricfg_buildinfo`, correspond to the first two lines printed by `gp` just before the Copyright message. The character string `paricfg_compiledate` is the date of compilation which appears on the next line. The character string `paricfg_mt_engine` is the name of the threading engine on the next line.

In the string `paricfg_buildinfo`, the substring `"%s"` needs to be substituted by the output of the function `pari_kernel_version`.

`const char * pari_kernel_version(void)`

`GEN pari_version()` returns the version number as a PARI object, a `t_VEC` with three `t_INT` and one `t_STR` components.

5.9.2 Miscellaneous.

`paricfg_datadir`: character string. The location of PARI's `datadir`.

`paricfg_gphelp`: character string. The name of an external help command for `??` (such as the `gphelp` script)

Chapter 6:
Arithmetic kernel: Level 0 and 1

6.1 Level 0 kernel (operations on ulongs).

6.1.1 Micro-kernel. The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need "global" `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER;  /* provides 'hiremainder' */
LOCAL_OVERFLOW;     /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong p` protected from overflows could read

```
while (pk < lim)
{
  LOCAL_HIREMAINDER;
  ...
  pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above 'global' variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bfffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG` $-$ 1 for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of (`hiremainder` $* 2^{\texttt{BITS\_IN\_LONG}}$) + `x` by `y` and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially `hiremainder` $\geq$ `y`.

`long hammingu(ulong x))` returns the Hamming weight of x, i.e. the number of nonzero bits in its binary expansion.

`long thuemorseu(ulong x)` return the parity of the number of nonzero bits in the binary expansion of x, that is, the Hamming weight modulo 2.

Obsolete routines. Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiftl(ulong x, ulong y)` returns x shifted left by y bits, i.e. x `<<` y, where we assume that $0 \leq y \leq$ `BITS_IN_LONG`. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `>>` (`BITS_IN_LONG` $- y$).

`ulong shiftlr(ulong x, ulong y)` returns x shifted right by y bits, i.e. x `>>` y, where we assume that $0 \leq y \leq$ `BITS_IN_LONG`. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `<<` (`BITS_IN_LONG` $- y$).

6.1.2 Modular kernel. The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let $m \geq 1$ be the modulus; all operands representing classes modulo m are assumed to belong to $[0, m-1]$. The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if x is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if x and y are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of x^{-1} mod $2^{\texttt{BITS\_IN\_LONG}}$, assuming x is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of $x + y$ modulo m.

`ulong Fl_neg(ulong x, ulong m)` returns the smallest nonnegative representative of $-x$ modulo m.

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of $x - y$ modulo m.

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in $]-m/2, m/2]$ of x modulo m. Assume $0 \leq x < m$ and `mo2` $= m >> 1$.

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of xy modulo m.

`ulong Fl_double(ulong x, ulong m)` returns $2x$ modulo m.

`ulong Fl_triple(ulong x, ulong m)` returns $3x$ modulo m.

`ulong Fl_halve(ulong x, ulong m)` returns z such that $2z = x$ modulo m assuming such z exists.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest nonnegative representative of x^2 modulo m.

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, return 0 (which is ambiguous if $m = 1$).

`ulong Fl_invgen(ulong x, ulong m, ulong *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m. We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of xy^{-1} modulo m. If y is not invertible mod m, raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest nonnegative representative of x^n modulo m.

`GEN Fl_powers(ulong x, long n, ulong p)` returns $[\mathtt{x}^0, \dots, \mathtt{x}^{\mathtt{n}}]$ modulo m, as a `t_VECSMALL`.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of `x` modulo `p` (smallest nonnegative representative). Assumes `p` to be prime, and `x` to be a square modulo `p`.

`ulong Fl_sqrtl(ulong x, ulong l, ulong p)` returns a l-the root of `x` modulo `p`. Assumes `p` to be prime and $p \equiv 1 \pmod{l}$, and `x` to be a l-th power modulo `p`.

`ulong Fl_sqrtn(ulong a, ulong n, ulong p, ulong *zn)` returns `ULONG_MAX` if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if `zn` is not `NULL` set it to a primitive m-th root of 1, $m = \gcd(p-1, n)$ allowing to compute all m solutions in $\mathbf{F}_p$ of the equation $x^n = a$. Assume that `p` is prime.

`ulong Fl_log(ulong a, ulong g, ulong ord, ulong p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the `Fp a`. It is assumed that `o` is a multiple of the order of `a`, 0 being allowed (no nontrivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in $0, 1, \dots p-1$.

`ulong nonsquare_Fl(ulong p)` return a quadratic nonresidue modulo p, assuming p is an odd prime. If p is 3 mod 4, return $p - 1$, else return the smallest (prime) nonresidue.

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`ulong pgener_Zl(ulong p)` returns the smallest positive integer which is a primitive root modulo p^k for all $k \geq 1$, assuming p is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`, `L` is an `Flv`.

`ulong factorial_Fl(long n, ulong p)` return $n!$ mod p.

6.1.3 Modular kernel with "precomputed inverse".

This is based on an algorithm by T. Grandlund and N. Möller in "Improved division by invariant integers" `https://gmplib.org/~tege/division-paper.pdf`.

In the following, we set $B =$ `BITS_IN_LONG`.

`ulong get_Fl_red(ulong p)` returns a pseudoinverse pi for p. Namely an integer $0 < pi < B$ such that, given $0 \leq x < B^2$ (by two long words), we can compute the Euclidean quotient and remainder of x modulo p by performing 2 multiplications and some additions. Precisely, once we set $q = 2^k p$ for the unique k such that $B/2 \leq q < B$, the pseudoinverse pi is equal to the Euclidean quotient of $B^2 - qB + B - 1$ by q. In particular $(pi + B)/B^2$ is very close to $1/q$.

Note that this algorithm is generally less efficient than ordinary quotient and remainders (`divll` or even `/` and `%`) when $0 \leq x < B$ and $p \leq B^{1/2}$ are small. High level functions below allow setting $pi = 0$ to cater for this possibility and avoid calling `get_Fl_red` for arguments where the standard algorithm is preferable.

`ulong divll_pre(ulong x, ulong p, ulong pi)` as divll, where pi is the pseudoinverse of p.

`ulong remll_pre(ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_1 2^B + u_0$ modulo p, assuming pi is the pseudoinverse of p. This function is faster if $u_1 < p$.

`ulong remlll_pre(ulong u2, ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_2 2^{2B} + u_1 2^B + u_0$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_sqr_pre(ulong x, ulong p, ulong pi)` returns x^2 modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_mul_pre(ulong x, ulong y, ulong p, ulong pi)` returns xy modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_addmul_pre(ulong a, ulong b, ulong c, ulong p, ulong pi)` returns $a+bc$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_addmulmul_pre(ulong a, ulong b, ulong c, ulong d, ulong p, ulong pi)` returns $ab + cd$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_powu_pre(ulong x, ulong n, ulong p, ulong pi)` returns x^n modulo p, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`GEN Fl_powers_pre(ulong x, long n, ulong p, ulong pi)` returns the vector (`t_VECSMALL`) $(x^0, \ldots, x^n)$, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_log_pre(ulong a, ulong g, ulong ord, ulong p, ulong pi)` as `Fl_log`, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrt_pre(ulong x, ulong p, ulong pi)` returns a square root of x modulo p, see `Fl_sqrt`. Assume `p` is prime. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtl_pre(ulong x, ulong l, ulong p, ulong pi)` returns a l-the root of `x` modulo `p`, assuming p prime, $p \equiv 1 \pmod{l}$, and `x` to be a l-th power modulo `p`. We assume pi is the

pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtn_pre(ulong x, ulong n, ulong p, ulong pi, ulong *zn)` See `Fl_sqrtn`, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_2gener_pre(ulong p, ulong pi)` return a generator of the 2-Sylow subgroup of $\mathbf{F}_p^*$, to be used in `Fl_sqrt_pre_i`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_2gener_pre_i(ulong ns, ulong p, ulong pi)` as `Fl_2gener_pre` where `ns` is a non-square modulo p.

`ulong Fl_sqrt_pre_i(ulong x, ulong s2, ulong p, ulong pi)` as `Fl_sqrt_pre` where `s2` is the element returned by `Fl_2gener_pre`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

6.1.4 Switching between Fl_xxx and standard operators.

Even though the `Fl_xxx` routines are efficient, they are slower than ordinary `long` operations, using the standard `+`, `%`, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

`int SMALL_ULONG(ulong p)` true if $2p^2 < 2^{\texttt{BITS\_IN\_LONG}}$. In that case, it is possible to use ordinary operators efficiently. If $p < 2^{\texttt{BITS\_IN\_LONG}}$, one may still use the `Fl_xxx` routines. Otherwise, one must use generic routines. For instance, the scalar product of the `GEN`s x and y mod p could be computed as follows.

```
long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
  GEN s = gen_0;
  for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
  return modii(s, p).
}
else
{
  ulong s = 0, pp = itou(p);
  x = ZV_to_Flv(x, pp);
  y = ZV_to_Flv(y, pp);
  if (SMALL_ULONG(pp))
  { /* very small */
    for (i = 1; i < l; i++)
    {
      s += x[i] * y[i];
      if (s & HIGHBIT) s %= pp;
    }
    s %= pp;
  }
  else
  { /* small */
```

```
    for (i = 1; i < l; i++)
      s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
  }
  return utoi(s);
}
```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

6.2 Level 1 kernel (operations on longs, integers and reals).

6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to -1.

`GEN cgetr(long prec)` allocates memory on the PARI stack for a `t_REAL` of precision `prec`, and initializes its first codeword. Identical to `cgetg(nbits2lg(prec), t_REAL)`.

`GEN cgetc(long prec)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of precision `prec`.

`GEN real_1(long prec)` create a `t_REAL` equal to 1 with precision `prec`.

`GEN real_m1(long prec)` create a `t_REAL` equal to -1 to `prec` words of accuracy.

`GEN real_0_expo(long e)` create a `t_REAL` equal to 0 with exponent e.

`GEN real_0(long prec)` is a shorthand for `real_0_expo(-prec)`. It allows to initialize real constants with `real_0`, `real_1`, ...`mplog2`, `mppi` ...in a uniform way.

`GEN int2n(long n)` creates a `t_INT` equal to `1<<n` (i.e 2^n if $n \geq 0$, and 0 otherwise).

`GEN int2u(ulong n)` creates a `t_INT` equal to 2^n.

`GEN int2um1(long n)` creates a `t_INT` equal to $2^n - 1$.

`GEN real2n(long n, long prec)` create a `t_REAL` equal to 2^n to `prec` words of accuracy.

`GEN real_m2n(long n, long prec)` create a `t_REAL` equal to -2^n to `prec` words of accuracy.

`GEN strtoi(char *s)` convert the character string `s` to a nonnegative `t_INT`. Decimal numbers, hexadecimal numbers prefixed by `0x` and binary numbers prefixed by `0b` are allowed. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

`GEN strtor(char *s, long prec)` convert the character string `s` to a nonnegative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.

6.2.2 Assignment. It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be useful in complicated garbage collecting scenarios. When the type and size of the PARI objects which appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore `avma`.

In this section, the `z` argument in the `z`-functions must be of type `t_INT` or `t_REAL`.

`int is_universal_constant(GEN x)` returns 1 if x is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

`void affii(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_INT` `z`.

`void affir(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affiz(GEN x, GEN z)` assigns `t_INT` `x` into `t_INT` or `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affsi(long s, GEN z)` assigns the `long` `s` into the `t_INT` `z`. Assumes that `lg(z)` > 2.

`void affsr(long s, GEN z)` assigns the `long` `s` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affsz(long s, GEN z)` assigns the `long` `s` into the `t_INT` or `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affui(ulong u, GEN z)` assigns the `ulong` `u` into the `t_INT` `z`. Assumes that `lg(z)` > 2.

`void affur(ulong u, GEN z)` assigns the `ulong` `u` into the `t_REAL` `z`. Assumes that `lg(z)` > 2.

`void affrr(GEN x, GEN z)` assigns the `t_REAL` `x` into the `t_REAL` `z`.

`void affgr(GEN x, GEN z)` assigns the real number (`t_INT`, `t_FRAC` or `t_REAL`) `x` into the `t_REAL` `z`.

`void affrr_fixlg(GEN y, GEN z)` a variant of `affrr`. First shorten z so that it is no longer than y, then assigns y to z. This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening z is not quite straightforward, since `setlg(z, ly)` would leave garbage on the stack, which garbage collection functions might later inspect. It is done using

`void fixlg(GEN z, long ly)` see `stackdummy` and the examples that follow.

6.2.3 Copy.

`GEN icopy(GEN x)` copy relevant words of the `t_INT` `x` on the stack: the length and effective length of the copy are equal.

`GEN rcopy(GEN x)` copy the `t_REAL` `x` on the stack.

`GEN leafcopy(GEN x)` copy the leaf `x` on the stack (works in particular for `t_INT`s and `t_REAL`s). Contrary to `icopy`, `leafcopy` preserves the original length of a `t_INT`. The obsolete form `GEN mpcopy(GEN x)` is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also `shallowcopy`.

`GEN leafcopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a leaf and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

`GEN icopyspec(GEN x, long nx)` copy the `nx` words `x[2]`, ..., `x[nx+1]` to make up a new `t_INT`. Set the sign to 1.

6.2.4 Conversions.

`GEN itor_lg(GEN x, long l)` converts the `t_INT x` to a `t_REAL` of length `l` and return the latter. Assumes that $\texttt{l} > 2$.

`GEN itor(GEN x, long prec)` converts the `t_INT x` to a `t_REAL` of `prec` and return the latter. Assumes that $\texttt{prec} > 0$.

`long itos(GEN x)` converts the `t_INT x` to a `long` if possible, otherwise raise an exception. We consider the conversion to be possible if and only if $|x| \leq$ `LONG_MAX`, i.e. $|x| < 2^{63}$ on a 64-bit architecture. Since the range is symmetric, the output of `itos` can safely be negated.

`long itos_or_0(GEN x)` converts the `t_INT x` to a `long` if possible, otherwise return 0.

`int is_bigint(GEN n)` true if `itos(n)` would give an error.

`ulong itou(GEN x)` converts the `t_INT |x|` to an `ulong` if possible, otherwise raise an exception. The conversion is possible if and only if $\texttt{lgefint}(x) \leq 3$.

`long itou_or_0(GEN x)` converts the `t_INT |x|` to an `ulong` if possible, otherwise return 0.

`GEN stoi(long s)` creates the `t_INT` corresponding to the `long s`.

`GEN stor_lg(long s, long l)` converts the `long s` into a `t_REAL` of length `l` and return the latter. Assumes that $\texttt{l} > 2$.

`GEN stor(long s, long prec)` converts the `long s` into a `t_REAL` of precision `prec` and return the latter. Assumes that $\texttt{prec} > 0$.

`GEN utoi(ulong s)` converts the `ulong s` into a `t_INT` and return the latter.

`GEN utoipos(ulong s)` converts the *nonzero* `ulong s` into a `t_INT` and return the latter.

`GEN utoineg(ulong s)` converts the *nonzero* `ulong s` into the `t_INT` $-s$ and return the latter.

`GEN utor_lg(ulong s, long l)` converts the `ulong s` into a `t_REAL` of length `l` and return the latter. Assumes that $\texttt{l} > 2$.

`GEN utor(ulong s, long prec)` converts the `ulong s` into a `t_REAL` of precision `prec` and return the latter. Assumes that $\texttt{prec} > 0$.

`GEN rtor_lg(GEN x, long l)` converts the `t_REAL x` to a `t_REAL` of length `l` and return the latter. If $\texttt{l} < \texttt{lg(x)}$, round properly. If $\texttt{l} > \texttt{lg(x)}$, pad with zeroes. Assumes that $\texttt{l} > 2$.

`GEN rtor(GEN x, long prec)` converts the `t_REAL x` to a `t_REAL` of precision `prec` and return the latter. If $\texttt{prec} < \texttt{realprec(x)}$, round properly. If $\texttt{prec} > \texttt{realprec(x)}$, pad with zeroes. Assumes that $\texttt{prec} > 0$.

The following functions are variants of `mkintn` with `unsigned long` arguments (instead of `unsigned int` for `mkintn`). As `mkintn`, they are mostly used to create large integer constants in a platform independent way:

`GEN uu32toi(ulong a, ulong b)` returns the `GEN` equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

`GEN uu32toineg(ulong a, ulong b)` returns the `GEN` equal to $-(2^{32}a+b)$, *assuming* that $a, b < 2^{32}$ and that one of a or b is positive. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

`GEN uutoi(ulong a, ulong b)` returns the `GEN` equal to $2^{\texttt{BITS\_IN\_LONG}}a + b$.

`GEN uutoineg(ulong a, ulong b)` returns the `GEN` equal to $-(2^{\texttt{BITS\_IN\_LONG}}a + b)$.

6.2.5 Integer parts. The following four functions implement the conversion from `t_REAL` to `t_INT` using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the `t_REAL` represents a range containing more than one integer.

`GEN ceilr(GEN x)` smallest integer larger or equal to the `t_REAL x` (i.e. the `ceil` function).

`GEN floorr(GEN x)` largest integer smaller or equal to the `t_REAL x` (i.e. the `floor` function).

`GEN roundr(GEN x)` rounds the `t_REAL x` to the nearest integer (towards $+\infty$ in case of tie).

`GEN truncr(GEN x)` truncates the `t_REAL x` (not the same as `floorr` if `x` is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a `t_INT`:

`GEN mpceil(GEN x)` as `ceilr` except that `x` may be a `t_INT`.

`GEN mpfloor(GEN x)` as `floorr` except that `x` may be a `t_INT`.

`GEN mpround(GEN x)` as `roundr` except that `x` may be a `t_INT`.

`GEN mptrunc(GEN x)` as `truncr` except that `x` may be a `t_INT`.

`GEN diviiround(GEN x, GEN y)` if `x` and `y` are `t_INT`s, returns the quotient `x/y` of `x` and `y`, rounded to the nearest integer. If `x/y` falls exactly halfway between two consecutive integers, then it is rounded towards $+\infty$ (as for `roundr`).

`GEN ceil_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the smallest integer which is larger than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.) Note that `gceil` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN floor_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the largest integer which is smaller than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.) Note that `gfloor` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN trunc_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.)

`GEN roundr_safe(GEN x)` rounds the `t_REAL x` to the nearest integer (towards $+\infty$). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

6.2.6 2-adic valuations and shifts.

`long vals(long s)` 2-adic valuation of the `long s`. Returns -1 if `s` is equal to 0.

`long vali(GEN x)` 2-adic valuation of the `t_INT x`. Returns -1 if `x` is equal to 0.

`GEN mpshift(GEN x, long n)` shifts the `t_INT` or `t_REAL x` by `n`. If `n` is positive, this is a left shift, i.e. multiplication by $2^{\mathtt{n}}$. If `n` is negative, it is a right shift by $-\mathtt{n}$, which amounts to the truncation of the quotient of `x` by $2^{-\mathtt{n}}$.

`GEN shifti(GEN x, long n)` shifts the `t_INT` x by n.

`GEN shiftr(GEN x, long n)` shifts the `t_REAL` x by n.

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL` x by n, in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL` x, returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL` x, returns the mantissa of $x2^n$ (disregards the exponent of x). Equivalent to

```
trunc2nr(x, n-expo(x)+realprec(x)-1)
```

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets `*e` to the exponent $\mathtt{bit\_accuracy}(\mathtt{lg}(\mathtt{z})) - 1 - \mathtt{expo}(z)$, so that $z = m/2^e$.

Low-level. In the following two functions, s(ource) and t(arget) need not be valid `GEN`s (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally s by $n > 0$ bits, storing the result in t. We assume that $m \leq M$ and only access $s[m], s[m+1], \dots s[M]$ (read) and likewise for t (write); we may have $s = t$ but more general overlaps are not allowed. The word f is concatenated to s to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$s[m], s[m+1], \dots s[M], f$$

left by n bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$f, s[m], s[m+1], \dots s[M]$$

right by n bits.

6.2.7 From t_INT to bits or digits in base 2^k and back.

GEN binary_zv(GEN x) given a t_INT x, return a t_VECSMALL of bits, from most significant to least significant.

GEN binary_2k(GEN x, long k) given a t_INT x, and $k > 0$, return a t_VEC of digits of x in base 2^k, as t_INTs, from most significant to least significant.

GEN binary_2k_nv(GEN x, long k) given a t_INT x, and $0 < k <$ BITS_IN_LONG, return a t_VECSMALL of digits of x in base 2^k, as ulongs, from most significant to least significant.

GEN bits_to_int(GEN x, long l) given a vector x of l bits (as a t_VECSMALL or even a pointer to a part of a larger vector, so not a proper GEN), return the integer $\sum_{i=1}^{l} x[i]2^{l-i}$, as a t_INT.

ulong bits_to_u(GEN v, long l) same as bits_to_int, where $l <$ BITS_IN_LONG, so we can return an ulong.

GEN fromdigitsu(GEN x, GEN B) given a t_VECSMALL x of length l and a t_INT B, return the integer $\sum_{i=1}^{l} x[i]B^{i-1}$, as a t_INT, where the x[i] are seen as unsigned integers.

GEN fromdigits_2k(GEN x, long k) converse of binary_2k; given a t_VEC x of length l and a positive long k, where each $x[i]$ is a t_INT with $0 \le x[i] < 2^k$, return the integer $\sum_{i=1}^{l} x[i]2^{k(l-i)}$, as a t_INT.

GEN nv_fromdigits_2k(GEN x, long k) as fromdigits_2k, but with x being a t_VECSMALL and each $x[i]$ being a ulong with $0 \le x[i] < 2^{\min\{k,\text{BITS\_IN\_LONG}\}}$. Here k may be any positive long, and the $x[i]$ are regarded as k-bit integers by truncating or extending with zeroes.

6.2.8 Integer valuation. For integers x and p, such that $x \neq 0$ and $|p| > 1$, we define $v_p(x)$ to be the largest integer exponent e such that p^e divides x. If p is prime, this is the ordinary valuation of x at p.

long Z_pvalrem(GEN x, GEN p, GEN *r) applied to t_INTs x $\neq 0$ and p, |p| > 1, returns $e := v_p(x)$ The quotient x/p^e is returned in *r. If |p| is a prime, *r is the prime-to-p part of x.

long Z_pval(GEN x, GEN p) as Z_pvalrem but only returns $v_p(x)$.

long Z_lvalrem(GEN x, ulong p, GEN *r) as Z_pvalrem, except that p is an ulong (p > 1).

long Z_lvalrem_stop(GEN *x, ulong p, int *stop) assume $x > 0$; returns $e := v_p(x)$ and replaces x by x/p^e. Set stop to 1 if the new value of x is $< p^2$ (and 0 otherwise). To be used when trial dividing x by successive primes: the stop condition is cheaply tested while testing whether p divides x (is the quotient less than p?), and allows to decide that n is prime if no prime $< p$ divides n. Not stack-clean.

long Z_lval(GEN x, ulong p) as Z_pval, except that p is an ulong (p > 1).

long u_lvalrem(ulong x, ulong p, ulong *r) as Z_pvalrem, except the inputs/outputs are now ulongs.

long u_lvalrem_stop(ulong *n, ulong p, int *stop) as Z_pvalrem_stop.

long u_pvalrem(ulong x, GEN p, ulong *r) as Z_pvalrem, except x and r are now ulongs.

long u_lval(ulong x, ulong p) as Z_pval, except the inputs are now ulongs.

long u_pval(ulong x, GEN p) as Z_pval, except x is now an ulong.

`long z_lval(long x, ulong p)` as `u_lval`, for signed `x`.

`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed `x`.

`long z_pval(long x, GEN p)` as `Z_pval`, except `x` is now a `long`.

`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except `x` is now a `long`.

`long factorial_lval(ulong n, ulong p)` returns $v_p(n!)$, assuming p is prime.

The following convenience functions generalize `Z_pval` and its variants to "containers" (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)` x being a `ZV` (a vector of `t_INT`s), return the min v of the valuations of its components and set `*r` to x/p^v. Infinite loop if x is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the "valuation".

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if p divides all components of x and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that `p` is an `ulong` (`p` > 1). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that `p` is an `ulong` (`p` > 1).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX` x (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX` x.

`long ZX_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_lvalrem`, a `ZX` x. This function is not stack-clean.

`long ZXX_pval(GEN x, GEN p)` as `ZX_pval`, for a `ZXX` x (a `t_POL` with `ZX` coefficients).

`long ZXX_pvalrem(GEN x, GEN p, GEN *r)` as `ZX_pvalrem`, for a `ZXX` x (a `t_POL` with `ZX` coefficients). This function is not stack-clean.

`long ZXV_pval(GEN x, GEN p)` as `ZV_pval`, for a `ZXV` x (a `t_VEC` with `ZX` coefficients).

`long ZXV_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZXV` x (a `t_VEC` with `ZX` coefficients). This function is not stack-clean.

`long ZX_lval(GEN x, ulong p)` as `ZX_pval`, except that `p` is an `ulong` (`p` > 1).

6.2.9 Generic unary operators. Let "*op*" be a unary operation among

- **neg**: negation ($-x$).
- **abs**: absolute value ($|x|$).
- **sqr**: square (x^2).

The names and prototypes of the low-level functions corresponding to *op* are as follows. The result is of the same type as `x`.

`GEN` *op*`i(GEN x)` creates the result of *op* applied to the `t_INT` `x`.

`GEN` *op*`r(GEN x)` creates the result of *op* applied to the `t_REAL` `x`.

`GEN mp`*op*`(GEN x)` creates the result of *op* applied to the `t_INT` or `t_REAL` `x`.

Complete list of available functions:

`GEN absi(GEN x)`, `GEN absr(GEN x)`, `GEN mpabs(GEN x)`

`GEN negi(GEN x)`, `GEN negr(GEN x)`, `GEN mpneg(GEN x)`

`GEN sqri(GEN x)`, `GEN sqrr(GEN x)`, `GEN mpsqr(GEN x)`

`GEN absi_shallow(GEN x)` x being a `t_INT`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `negi(`x`)` otherwise.

`GEN mpabs_shallow(GEN x)` x being a `t_INT` or a `t_REAL`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `mpneg(`x`)` otherwise.

Some miscellaneous routines:

`GEN sqrs(long x)` returns x^2.

`GEN sqru(ulong x)` returns x^2.

6.2.10 Comparison operators.

`int cmpss(long s, long t)` compares the `long` `s` to the `t_long` `t`.

`int cmpuu(ulong u, ulong v)` compares the `ulong` `u` to the `t_ulong` `v`.

`long minss(long x, long y)`

`ulong minuu(ulong x, ulong y)`

`double mindd(double x, double y)` returns the `min` of x and y.

`long maxss(long x, long y)`

`ulong maxuu(ulong x, ulong y)`

`double maxdd(double x, double y)` returns the `max` of x and y.

`int mpcmp(GEN x, GEN y)` compares the `t_INT` or `t_REAL` `x` to the `t_INT` or `t_REAL` `y`. The result is the sign of `x` $-$ `y`.

`int cmpii(GEN x, GEN y)` compares the `t_INT` `x` to the `t_INT` `y`.

`int cmpir(GEN x, GEN y)` compares the `t_INT` `x` to the `t_REAL` `y`.

`int cmpis(GEN x, long s)` compares the `t_INT` `x` to the `long` `s`.

`int cmpiu(GEN x, ulong s)` compares the `t_INT` `x` to the `ulong` `s`.

`int cmpsi(long s, GEN x)` compares the `long` `s` to the `t_INT` `x`.

`int cmpui(ulong s, GEN x)` compares the `ulong` `s` to the `t_INT` `x`.

`int cmpsr(long s, GEN x)` compares the `long` `s` to the `t_REAL` `x`.

`int cmpri(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_INT` `y`.

`int cmprr(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_REAL` `y`.

`int cmprs(GEN x, long s)` compares the `t_REAL` `x` to the `long` `s`.

`int equalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if $x = y$, 0 otherwise.

`int equalrr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is 1 if $x = y$, 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a nonzero real; two nonzero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.

`int equalis(GEN x, long s)` compare the `t_INT` `x` and the `long` `s`. The result is 1 if $x = y$, 0 otherwise.

`int equalsi(long s, GEN x)`

`int equaliu(GEN x, ulong s)` compare the `t_INT` `x` and the `ulong` `s`. The result is 1 if $x = y$, 0 otherwise.

`int equalui(ulong s, GEN x)`

The remaining comparison operators disregard the sign of their operands

`int absequaliu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if $|x| = y$, 0 otherwise. This is marginally more efficient than `equalis` even when `x` is known to be nonnegative.

`int absequalui(ulong u, GEN x)`

`int abscmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `u`.

`int abscmpui(ulong u, GEN x)`

`int abscmpii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of $|x| - |y|$.

`int absequalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if $|x| = |y|$, 0 otherwise.

`int abscmprr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of $|x| - |y|$.

`int absrnz_equal2n(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to $\pm 2^e$ for some integer e.

`int absrnz_equal1(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to ± 1.

6.2.11 Generic binary operators. The operators in this section have arguments of C-type `GEN`, `long`, and `ulong`, and only `t_INT` and `t_REAL` `GEN`s are allowed. We say an argument is a real type if it is a `t_REAL` `GEN`, and an integer type otherwise. The result is always a `t_REAL` unless both `x` and `y` are integer types.

Let "*op*" be a binary operation among

• **add**: addition (`x + y`).

• **sub**: subtraction (`x - y`).

• **mul**: multiplication (`x * y`).

• **div**: division (`x / y`). In the case where `x` and `y` are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend `x`. It is the ordinary division otherwise. A division-by-0 error occurs if `y` is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a `t_INT`:

• **rem**: remainder ("`x % y`"). The result is the Euclidean remainder corresponding to `div`, i.e. its sign is that of the dividend `x`.

• **mod**: true remainder (`x % y`). The result is the true Euclidean remainder, i.e. nonnegative and less than the absolute value of `y`.

Important technical note. The rules given above fixing the output type (to `t_REAL` unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI's generic functions, such as `gmul` or `gdiv` for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if x is a `t_INT` and y a `t_REAL`, then `gmul(x,y)` is always the same as `mulir(x,y)`. The exception is $x = 0$, in that case we can deduce that the result is an exact 0, so `gmul` returns `gen_0`, while `mulir` returns a `t_REAL` 0. Specifically, the one resulting from the conversion of `gen_0` to a `t_REAL` of precision `precision(y)`, multiplied by y; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, such as being nonzero).

The names and prototypes of the low-level functions corresponding to *op* are as follows.

`GEN mp`*op*`(GEN x, GEN y)` applies *op* to the `t_INT` or `t_REAL` `x` and `y`. The result is a `t_INT` if both inputs are integers and a `t_REAL` otherwise.

Warning about division. the `mpdiv` function is obsolete and should not be used: it is only sensible when one argument is a `t_REAL`. Indeed, if both inputs are `t_INT` : we can't return x/y as a `t_FRAC` (not an allowed type here) or as a `t_REAL` since its accuracy can't be determined from the inputs; finally Euclidean division semantics $x = qy + r$ is not the desired one. Thus `mpdiv` will error out in this case of `t_INT` arguments, to avoid unexpected behaviour. Instead we have `divrmp`(x, y) when x is a `t_REAL` and y is a `t_INT` or `t_REAL`; and `divmpr`(x, y) when y is a `t_REAL`.

`GEN` *op*`si(long s, GEN x)` applies *op* to the `long s` and the `t_INT x`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`sr(long s, GEN x)` applies *op* to the `long s` and the `t_REAL x`.

`GEN` *op*`ss(long s, long t)` applies *op* to the longs `s` and `t`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ii(GEN x, GEN y)` applies *op* to the `t_INT`s `x` and `y`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ir(GEN x, GEN y)` applies *op* to the `t_INT x` and the `t_REAL y`.

`GEN` *op*`is(GEN x, long s)` applies *op* to the `t_INT x` and the `long s`. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN` *op*`ri(GEN x, GEN y)` applies *op* to the `t_REAL x` and the `t_INT y`.

`GEN` *op*`rr(GEN x, GEN y)` applies *op* to the `t_REAL`s `x` and `y`.

`GEN` *op*`rs(GEN x, long s)` applies *op* to the `t_REAL x` and the `long s`.

Some miscellaneous routines:

`long expu(ulong x)` assuming $x > 0$, returns the binary exponent of the real number equal to x. This is a special case of `gexpo`.

`GEN adduu(ulong x, ulong y)`

`GEN addiu(GEN x, ulong y)`

`GEN addui(ulong x, GEN y)` adds `x` and `y`.

`GEN subuu(ulong x, ulong y)`

`GEN subiu(GEN x, ulong y)`

`GEN subui(ulong x, GEN y)` subtracts `x` by `y`.

`GEN muluu(ulong x, ulong y)` multiplies `x` by `y`.

`ulong umuluu_le(ulong x, ulong y, ulong n)` multiplies `x` by `y`. Return xy if $xy \leq n$ and 0 otherwise (in particular if xy does not fit in an `ulong`).

`ulong umuluu_or_0(ulong x, ulong y)` multiplies `x` by `y`. Return 0 if xy does not fit in an `ulong`.

`GEN mului(ulong x, GEN y)` multiplies `x` by `y`.

`GEN muluui(ulong x, ulong y, GEN z)` return xyz.

`GEN muliu(GEN x, ulong y)` multiplies `x` by `y`.

`void addumului(ulong a, ulong b, GEN x)` return $a + b|X|$.

`GEN addmuliu(GEN x, GEN y, ulong u)` returns $x + yu$.

`GEN addmulii(GEN x, GEN y, GEN z)` returns $x + yz$.

`GEN addmulii_inplace(GEN x, GEN y, GEN z)` returns $x + yz$, but returns x itself and not a copy if $yz = 0$. Shallow function.

`GEN addmuliu_inplace(GEN x, GEN y, ulong u)` returns $x + yu$, but returns x itself and not a copy if $yu = 0$. Shallow function.

`GEN submuliu_inplace(GEN x, GEN y, ulong u)` returns $x - yu$, but returns x itself and not a copy if $yu = 0$. Shallow function.

`GEN lincombii(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$.

`GEN mulsubii(GEN y, GEN z, GEN x)` returns $yz - x$.

`GEN submulii(GEN x, GEN y, GEN z)` returns $x - yz$.

`GEN submuliu(GEN x, GEN y, ulong u)` returns $x - yu$.

`GEN mulu_interval(ulong a, ulong b)` returns $a(a + 1) \cdots b$, assuming that $a \leq b$.

`GEN mulu_interval_step(ulong a, ulong b, ulong s)` returns the product of all integers in $[a, b]$ congruent to a modulo s. Assume $a \leq b$ and $s > 0$;

`GEN muls_interval(long a, long b)` returns $a(a + 1) \cdots b$, assuming that $a \leq b$.

`GEN invr(GEN x)` returns the inverse of the nonzero `t_REAL` x.

`GEN truedivii(GEN x, GEN y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN truedivis(GEN x, long y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN truedivsi(long x, GEN y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN centermodii(GEN x, GEN y, GEN y2)`, given `t_INT`s `x`, `y`, returns z congruent to `x` modulo `y`, such that $-\mathtt{y}/2 \leq z < \mathtt{y}/2$. The function requires an extra argument `y2`, such that `y2 = shifti(y, -1)`. (In most cases, `y` is constant for many reductions and `y2` need only be computed once.)

`GEN remi2n(GEN x, long n)` returns `x` mod 2^n.

`GEN addii_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT`s x and y as if their signs were `sx` and `sy`.

`GEN addir_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT` x and the `t_REAL` y as if their signs were `sx` and `sy`.

`GEN addrr_sign(GEN x, long sx, GEN y, long sy)` add the `t_REAL`s x and y as if their signs were `sx` and `sy`.

`GEN addsi_sign(long x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

`GEN addui_sign(ulong x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

6.2.12 Exact division and divisibility.

`GEN diviiexact(GEN x, GEN y)` returns the Euclidean quotient `x/y`, assuming `y` divides `x`. Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

`GEN diviuexact(GEN x, ulong y)` returns the Euclidean quotient `x/y`, assuming `y` divides `x` and `y` is nonzero.

`GEN diviuuexact(GEN x, ulong y, ulong z)` returns the Euclidean quotient $x/(yz)$, assuming yz divides x and $yz \neq 0$.

The following routines return 1 (true) if `y` divides `x`, and 0 otherwise. All `GEN` are assumed to be `t_INT`s:

`int dvdii(GEN x, GEN y)`, `int dvdis(GEN x, long y)`, `int dvdiu(GEN x, ulong y)`,

`int dvdsi(long x, GEN y)`, `int dvdui(ulong x, GEN y)`.

The following routines return 1 (true) if `y` divides `x`, and in that case assign the quotient to `z`; otherwise they return 0. All `GEN` are assumed to be `t_INT`s:

`int dvdiiz(GEN x, GEN y, GEN z)`, `int dvdisz(GEN x, long y, GEN z)`.

`int dvdiuz(GEN x, ulong y, GEN z)` if `y` divides `x`, assigns the quotient |x|/y to `z` and returns 1 (true), otherwise returns 0 (false).

6.2.13 Division with integral operands and `t_REAL` result.

`GEN rdivii_lg(GEN x, GEN y, long l)`, assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_REAL` of length `l` > 2.

`GEN rdivii(GEN x, GEN y, long prec)`, assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec` > 0.

`GEN rdiviiz(GEN x, GEN y, GEN z)`, assuming x and y are both of type `t_INT`, and z is a `t_REAL`, assign the quotient x/y to z.

`GEN rdivis(GEN x, long y, long prec)`, assuming `x` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivsi(long x, GEN y, long prec)`, assuming `y` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivss(long x, long y, long prec)`, return the quotient x/y as a `t_REAL` of precision `prec`.

6.2.14 Division with remainder. The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the nonnegative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All `GEN`s, whether returned directly or through a pointer, are created on the stack.

`GEN dvmdii(GEN x, GEN y, GEN *r)` returns the Euclidean quotient of the `t_INT` `x` by a `t_INT` `y` and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend `x`. If the remainder is 0 set `r = gen_0`.

Analogous routines `dvmdis`, `dvmdsi`, `dvmdss` are available, where `s` denotes a `long` argument. But the following routines are in general more flexible:

`long sdivss_rem(long s, long t, long *r)` computes the Euclidean quotient and remainder of the longs `s` and `t`. Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend `s`, and has strictly smaller absolute value than `t`.

`long sdivsi_rem(long s, GEN x, long *r)` computes the Euclidean quotient and remainder of the `long s` by the `t_INT x`. As `sdivss_rem` otherwise.

`long sdivsi(long s, GEN x)` as `sdivsi_rem`, without remainder.

`GEN divis_rem(GEN x, long s, long *r)` computes the Euclidean quotient and remainder of the `t_INT x` by the `long s`. As `sdivss_rem` otherwise.

`GEN absdiviu_rem(GEN x, ulong s, ulong *r)` computes the Euclidean quotient and remainder of *absolute value* of the `t_INT x` by the `ulong s`. As `sdivss_rem` otherwise.

`ulong uabsdiviu_rem(GEN n, ulong d, ulong *r)` as `absdiviu_rem`, assuming that $|n|/d$ fits into an `ulong`.

`ulong uabsdivui_rem(ulong x, GEN y, ulong *rem)` computes the Euclidean quotient and remainder of x by $|y|$. As `sdivss_rem` otherwise.

`ulong udivuu_rem(ulong x, ulong y, ulong *rem)` computes the Euclidean quotient and remainder of x by y. As `sdivss_rem` otherwise.

`ulong ceildivuu(ulong x, ulong y)` return the ceiling of x/y.

`GEN divsi_rem(long s, GEN y, long *r)` computes the Euclidean quotient and remainder of the `long s` by the `GEN y`. As `sdivss_rem` otherwise.

`GEN divss_rem(long x, long y, long *r)` computes the Euclidean quotient and remainder of the `long x` by the `long y`. As `sdivss_rem` otherwise.

`GEN truedvmdii(GEN x, GEN y, GEN *r)`, as `dvmdii` but with a nonnegative remainder.

`GEN truedvmdis(GEN x, long y, GEN *z)`, as `dvmdis` but with a nonnegative remainder.

`GEN truedvmdsi(long x, GEN y, GEN *z)`, as `dvmdsi` but with a nonnegative remainder.

6.2.15 Modulo to longs. The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT x` by the `long y`. This is the nonnegative remainder, not the one whose sign is the sign of `x` as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the `long x` by a `long y`.

`ulong umodsu(long x, ulong y)` computes the true Euclidean remainder of the `long x` by a `ulong y`.

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT x` by the `ulong y`.

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the `ulong x` by the `t_INT |y|`.

The routine `smodsi` does not exist, since it would not always be defined: for a *negative* `x`, if the quotient is ± 1, the result `x + |y|` would in general not fit into a `long`. Use either `umodui` or `modsi`.

These functions directly access the binary data and are thus much faster than the generic modulo functions:

`int mpodd(GEN x)` which is 1 if `x` is odd, and 0 otherwise.

`ulong Mod2(GEN x)`

`ulong Mod4(GEN x)`

`ulong Mod8(GEN x)`

`ulong Mod16(GEN x)`

`ulong Mod32(GEN x)`

`ulong Mod64(GEN x)` give the residue class of x modulo the corresponding power of 2.

`ulong umodi2n(GEN x, long n)` give the residue class of x modulo 2^n, $0 \leq n < BITS\_IN\_LONG$.

The following functions assume that $x \neq 0$ and in fact disregard the sign of x. There are about 10% faster than the safer variants above:

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of $|x|$ modulo the corresponding power of 2, for *nonzero* `x`. As well,

`ulong mod2BIL(GEN x)` returns the least significant word of $|x|$, still assuming that $x \neq 0$.

6.2.16 Powering, Square root.

`GEN powii(GEN x, GEN n)`, assumes x and n are `t_INT`s and returns x^n.

`GEN powuu(ulong x, ulong n)`, returns x^n.

`GEN powiu(GEN x, ulong n)`, assumes x is a `t_INT` and returns x^n.

`GEN powis(GEN x, long n)`, assumes x is a `t_INT` and returns x^n (possibly a `t_FRAC` if $n < 0$).

`GEN powrs(GEN x, long n)`, assumes x is a `t_REAL` and returns x^n. This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if $n = 0$. Note that the generic function `gpowgs(x,0)` would return `gen_1`, see the technical note in Section 6.2.11.

`GEN powru(GEN x, ulong n)`, assumes x is a `t_REAL` and returns x^n (always a `t_REAL`, even if $n = 0$).

`GEN powersr(GEN e, long n)`. Given a `t_REAL` e, return the vector v of all e^i, $0 \leq i \leq n$, where $v[i] = e^{i-1}$.

`GEN powrshalf(GEN x, long n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powruhalf(GEN x, ulong n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powrfrac(GEN x, long n, long d)`, assumes x is a `t_REAL` and returns $x^{n/d}$ (always a `t_REAL`, even if $n = 0$).

`GEN powIs(long n)` returns $I^n \in \{1, I, -1, -I\}$ (`t_INT` for even n, `t_COMPLEX` otherwise).

`ulong upowuu(ulong x, ulong n)`, returns x^n when $< 2^{\texttt{BITS\_IN\_LONG}}$, and 0 otherwise (overflow).

`ulong upowers(ulong x, long n)`, returns $[1, x, \ldots, x^n]$ as a `t_VECSMALL`. Assume there is no overflow.

`GEN sqrtremi(GEN N, GEN *r)`, returns the integer square root S of the nonnegative `t_INT N` (rounded towards 0) and puts the remainder R into `*r`. Precisely, $N = S^2 + R$ with $0 \le R \le 2S$. If `r` is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set `R = gen_0`.

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann ("Karatsuba Square Root", INRIA Research Report 3805 (1999)).

`GEN sqrti(GEN N)`, returns the integer square root S of the nonnegative `t_INT N` (rounded towards 0). This is identical to `sqrtremi(N, NULL)`.

`long logintall(GEN B, GEN y, GEN *ptq)` returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers. If `ptq` is not `NULL`, set it to y^e. (Analogous to `logint0`, whithout sanity checks.)

`ulong ulogintall(ulong B, ulong y, ulong *ptq)` as `logintall` for `ulong` arguments.

`long logint(GEN B, GEN y)` returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers.

`ulong ulogint(ulong B, ulong y)` as `logint` for `ulong` arguments.

`GEN vecpowuu(long N, ulong a)` return the vector of n^a, $n = 1, \ldots, N$. Not stack-clean.

`GEN vecpowug(long N, GEN a, long prec)` return the vector of n^a, $n = 1, \ldots, N$, where the powers are computed at precision `prec`. Not stack-clean.

6.2.17 GCD, extended GCD and LCM.

`long cgcd(long x, long y)` returns the GCD of `x` and `y`.

`ulong ugcd(ulong x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdiu(GEN x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdui(ulong x, GEN y)` returns the GCD of `x` and `y`.

`GEN coprimes_zv(ulong N)` return a `t_VECSMALL` T with N entries such that $T[i] = 1$ iff $(i, N) = 1$ and 0 otherwise.

`long clcm(long x, long y)` returns the LCM of `x` and `y`, provided it fits into a `long`. Silently overflows otherwise.

`ulong ulcm(ulong x, ulong y)` returns the LCM of `x` and `y`, provided it fits into an `ulong`. Silently overflows otherwise.

`GEN gcdii(GEN x, GEN y)`, returns the GCD of the `t_INT`s `x` and `y`.

`GEN lcmii(GEN x, GEN y)`, returns the LCM of the `t_INT`s `x` and `y`.

`GEN bezout(GEN a, GEN b, GEN *u, GEN *v)`, returns the GCD d of `t_INT`s `a` and `b` and sets `u`, `v` to the Bezout coefficients such that `au` + `bv` $= d$.

`long cbezout(long a, long b, long *u, long *v)`, returns the GCD d of `a` and `b` and sets `u`, `v` to the Bezout coefficients such that `au` + `bv` $= d$.

`GEN halfgcdii(GEN x, GEN y)` assuming `x` and `y` are `t_INT`s, returns a 2-components `t_VEC` $[M, V]$ where M is a 2×2 `t_MAT` and V a 2-component `t_COL`, both with `t_INT` entries, such that $M * [x, y]$ $==$ V and such that if $V = [a, b]$, then $a \geq \left\lceil \sqrt{\max(|x|, |y|)} \right\rceil > b$.

`GEN ZV_extgcd(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the GCD of the $A[i]$ and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [0, \ldots, 0, D]$.

`GEN ZV_lcm(GEN v)` given a vector v of integers returns the LCM of its entries.

`GEN ZV_snf_gcd(GEN v, GEN N)` given a vector v of integers and a positive integer N, return the vector whose entries are the gcds $(v[i], N)$. Use case: if v gives the cyclic components for some Abelian group G of finite type, then this returns the structure of the finite groupe G/G^N.

6.2.18 Continued fractions and convergents.

`GEN ZV_allpnqn(GEN x)` given $x = [a_0, ..., a_n]$ a continued fraction from `gboundcf`, $n \geq 0$, return all convergents as $[P, Q]$, where $P = [p_0, \ldots, p_n]$ and $Q = [q_0, \ldots, q_n]$.

6.2.19 Pseudo-random integers. These routine return pseudo-random integers uniformly distributed in some interval. The all use the same underlying generator which can be seeded and restarted using `getrand` and `setrand`.

`void setrand(GEN seed)` reseeds the random number generator using the seed n. The seed is either a technical array output by `getrand` or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

`GEN getrand(void)` returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type `t_VECSMALL`), and can only be used as an argument to `setrand`.

`ulong pari_rand(void)` returns a random $0 \leq x < 2^{\texttt{BITS\_IN\_LONG}}$.

`long random_bits(long k)` returns a random $0 \leq x < 2^k$. Assumes that $0 \leq k \leq$ `BITS_IN_LONG`.

`ulong random_Fl(ulong p)` returns a pseudo-random integer in $0, 1, \ldots p - 1$.

`GEN randomi(GEN n)` returns a random `t_INT` between 0 and `n` $- 1$.

`GEN randomr(long prec)` returns a random `t_REAL` in $[0, 1[$, with precision `prec`.

6.2.20 Modular operations. In this subsection, all GENs are t_INT.

GEN Fp_red(GEN a, GEN m) returns a modulo m (smallest nonnegative residue). (This is identical to modii).

GEN Fp_neg(GEN a, GEN m) returns $-$a modulo m (smallest nonnegative residue).

GEN Fp_add(GEN a, GEN b, GEN m) returns the sum of a and b modulo m (smallest nonnegative residue).

GEN Fp_sub(GEN a, GEN b, GEN m) returns the difference of a and b modulo m (smallest nonnegative residue).

GEN Fp_center(GEN a, GEN p, GEN pov2) assuming that pov2 is shifti(p,-1) and that $-p/2 < a < p$, returns the representative of a in the symmetric residue system $]-p/2, p/2]$.

GEN Fp_center_i(GEN a, GEN p, GEN pov2) shallow internal variant of Fp_center: when a is already in the proper interval, it is returned as is, without a copy.

GEN Fp_mul(GEN a, GEN b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

GEN Fp_addmul(GEN x, GEN y, GEN z, GEN p) returns $x + yz$.

GEN Fp_mulu(GEN a, ulong b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

GEN Fp_muls(GEN a, long b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

GEN Fp_halve(GEN x, GEN m) returns z such that $2z = x$ modulo m assuming such z exists. Assume that $0 \leq x < m$. Not stack-clean, but suitable for gc_upto.

GEN Fp_double(GEN x, GEN m) return $2x$ modulo m. Assume that $0 \leq x < m$. Not stack-clean, but suitable for gc_upto.

GEN Fp_sqr(GEN a, GEN m) returns $\mathtt{a}^2$ modulo m (smallest nonnegative residue).

ulong Fp_powu(GEN x, ulong n, GEN m) raises x to the n-th power modulo m (smallest nonnegative residue). Not stack-clean, but suitable for gc_upto.

ulong Fp_pows(GEN x, long n, GEN m) raises x to the n-th power modulo m (smallest nonnegative residue). A negative n is allowed Not stack-clean, but suitable for gc_upto.

GEN Fp_pow(GEN x, GEN n, GEN m) returns $\mathtt{x}^{\mathtt{n}}$ modulo m (smallest nonnegative residue).

GEN Fp_pow_init(GEN x, GEN n, long k, GEN p) Return a table R that can be used with Fp_pow_table to compute the powers of x up to n. The table is of size $2^k \log_2(n)$.

GEN Fp_pow_table(GEN R, GEN n, GEN p) return x^n, where R is given by Fp_pow_init(x,m,k,p) for some integer $m \geq n$.

GEN Fp_powers(GEN x, long n, GEN m) returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ modulo m as a t_VEC (smallest nonnegative residue).

GEN Fp_inv(GEN a, GEN m) returns an inverse of a modulo m (smallest nonnegative residue). Raise an error if a is not invertible.

GEN Fp_invsafe(GEN a, GEN m) as Fp_inv, but return NULL if a is not invertible.

`GEN Fp_invgen(GEN x, GEN m, GEN *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m. We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`GEN FpV_prod(GEN x, GEN p)` returns the product of the components of x.

`GEN FpV_inv(GEN x, GEN m)` x being a vector of `t_INT`s, return the vector of inverses of the $x[i]$ mod m. The routine uses Montgomery's trick, and involves a single inversion mod m, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ integers mod m are left on stack, besides the N in the result.

`GEN Fp_div(GEN a, GEN b, GEN m)` returns the quotient of `a` by `b` modulo `m` (smallest nonnegative residue). Raise an error if `b` is not invertible.

`GEN Fp_divu(GEN a, ulong b, GEN m)` returns the quotient of `a` by `b` modulo `m` (smallest nonnegative residue). Raise an error if `b` is not invertible.

`int invmod(GEN a, GEN m, GEN *g)`, return 1 if `a` modulo `m` is invertible, else return 0 and set `g` $= \gcd($`a`$,$`m`$)$.

In the following three functions the integer parameter `ord` can be given either as a positive `t_INT` N, or as its factorization matrix faN, or as a pair $[N, faN]$. The parameter may be omitted by setting it to `NULL` (the value is then $p-1$).

`GEN Fp_log(GEN a, GEN g, GEN ord, GEN p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`GEN Fp_order(GEN a, GEN ord, GEN p)` returns the order of the `Fp a`. Assume that `ord` is a multiple of the order of `a`.

`GEN Fp_factored_order(GEN a, GEN ord, GEN p)` returns $[o, F]$, where o is the multiplicative order of the `Fp` a in $\mathbf{F}_p^*$, and F is the factorization of o. Assume that `ord` is a multiple of the order of `a`.

`int Fp_issquare(GEN x, GEN p)` returns 1 if `x` is a square modulo `p`, and 0 otherwise. Assume that `p` is a prime.

`int Fp_ispower(GEN x, GEN n, GEN p)` returns 1 if `x` is an n-th power modulo `p`, and 0 otherwise. Assume that `p` is a prime.

`GEN Fp_sqrt(GEN x, GEN p)` returns a square root of `x` modulo `p` (the smallest nonnegative residue), where `x`, `p` are `t_INT`s, and `p` is assumed to be prime. Return `NULL` if `x` is not a quadratic residue modulo `p`.

`GEN Fp_2gener(GEN p)` return a generator of the 2-Sylow subgroup of $\mathbf{F}_p^*$. To use with `Fp_sqrt_i`. Assume that `p` is prime; may return `NULL` if `p` is proven to be composite.

`GEN Fp_2gener_i(GEN ns, GEN p)` as `Fp_2gener`, where `ns` is a non-square modulo p.

`GEN Fp_sqrt_i(GEN x, GEN s2, GEN p)` as `Fp_sqrt` where `s2` is the element returned by `Fp_2gener`.

`GEN Fp_sqrtn(GEN a, GEN n, GEN p, GEN *zn)` returns `NULL` if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if `zn` is not `NULL` set it to a primitive m-th root of 1, $m = \gcd(p-1, n)$ allowing to compute all m solutions in $\mathbf{F}_p$ of the equation $x^n = a$. Assume that `p` is prime.

`GEN Zn_sqrt(GEN x, GEN n)` returns one of the square roots of `x` modulo `n` (possibly not prime), where `x` is a `t_INT` and `n` is either a `t_INT` or is given by its factorization matrix. Return `NULL` if no such square root exist.

`GEN Zn_sqrtall(GEN x, GEN n)` returns all the square roots of `x` modulo `n` (possibly not prime), as a `t_VEC` of representatives, where `x` is a `t_INT` and `n` is either a `t_INT` or is given by its factorization matrix. Return `NULL` if no such square root exist.

`GEN Zn_quad_roots(GEN N, GEN B, GEN C)` solves the equation X^2+BX+C modulo N. Return `NULL` if there are no solutions. Else returns $[M, v]$ where $M \mid N$ and the `FpV` v of distinct integers (reduced, implicitly modulo M) is such that x modulo N is a solution to the equation if and only if x modulo M belongs to v. If the discriminant B^2-4C is coprime to N, we have $M = N$ but in general M can be a strict divisor of N.

`long kross(long x, long y)` returns the Kronecker symbol $(x|y)$, i.e.-1, 0 or 1. If `y` is an odd prime, this is the Legendre symbol. (Contrary to `krouu`, `kross` also supports `y` $= 0$)

`long krouu(ulong x, ulong y)` returns the Kronecker symbol $(x|y)$, i.e. -1, 0 or 1. Assumes `y` is nonzero. If `y` is an odd prime, this is the Legendre symbol.

`long krois(GEN x, long y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and `long y`. As `kross` otherwise.

`long kroiu(GEN x, ulong y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and nonzero `ulong y`. As `krouu` otherwise.

`long krosi(long x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT y`. As `kross` otherwise.

`long kroui(ulong x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT y`. As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol $(x|y)$ of `t_INT`s x and `y`. As `kross` otherwise.

`GEN factorial_Fp(long n, GEN p)` return $n!$ mod p.

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo p^k, $k > 1$, assuming `p` is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT` x is a p-adic square, 0 otherwise. Assume that `p` is prime.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT` x is a square modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise.

`long Zn_ispower(GEN x, GEN n, GEN K, GEN *py)` returns 1 if `t_INT` x is a K-th power modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise. If `py` is not `NULL`, set it to y such that $y^K = x$ modulo n.

`GEN pgener_Fp_local(GEN p, GEN L)`, `L` being a vector of primes dividing $p-1$, returns the smallest integer $x > 1$ which is a generator of the ℓ-Sylow of $\mathbf{F}_p^*$ for every ℓ in `L`. In other words, $x^{(p-1)/\ell} \neq 1$ for all such ℓ. In particular, returns `pgener_Fp(p)` if `L` contains all primes dividing $p-1$. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square. Assume that `p` is prime.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive n-th root modulo the prime p.

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive n-th root modulo the prime p.

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive n-th root modulo the prime p.

6.2.21 Extending functions to vector inputs.

The following functions apply f to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if x is a `t_VEC`, return a `t_VEC` whose components are the $f(x[i])$.

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi( f() )`.

`GEN map_proto_GL(GEN (*f)(GEN, long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN, long), GEN x, long y)`

In the last function, f implements an associative binary operator, which we extend naturally to an n-ary operator f_n for any n: by convention, $f_0() = 1$, $f_1(x) = x$, and

$$f_n(x_1, \dots, x_n) = f(f_{n-1}(x_1, \dots, x_{n-1}), x_n)),$$

for $n \geq 2$.

`GEN gassoc_proto(GEN (*f)(GEN, GEN), GEN x, GEN y)` If y is not `NULL`, return $f(x, y)$. Otherwise, x must be of vector type, and we return the result of f applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \texttt{NULL}), f(x_2, \texttt{NULL})),$$

where x_1, x_2 are the two halves of x.

6.2.22 Miscellaneous arithmetic functions.

`long bigomegau(ulong n)` returns the number of prime divisors of $n > 0$, counted with multiplicity.

`ulong coreu(ulong n)`, unique squarefree integer d dividing n such that n/d is a square.

`ulong coreu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong corediscs(long d, ulong *pt_f)`, d (possibly negative) being congruent to 0 or 1 modulo 4, return the fundamental discriminant D such that $d = D * f^2$ and set `*pt_f` to f (if `*pt_f` not `NULL`).

`GEN coredisc2_fact(GEN fa, long s, GEN *pP, GEN *pE)` let d be an integer congruent to 0 or 1 mod 4. Return $D =$ `coredisc`(d) assuming that `fa` is the factorization of $|d|$ and $sd > 0$ (s is the sign of d). Set `*pP` and `*pE` to the factorization of the conductor f such that $d = Df^2$, where P is a `t_VEC` of primes and E a `t_VECSMALL` of exponents.

`ulong coredisc2u_fact(GEN fa, long s, GEN *pP, GEN *pE)` let d be an integer congruent to 0 or 1 mod 4 whose absolute value fits in an `ulong`. Return the absolute value of $D =$ `corediscs`(d) assuming that `fa` is the factorization of $|d|$ and $sd > 0$ (s is the sign of d and D). Set `*pP` and `*pE`

to the factorization of the conductor f such that $d = Df^2$, where P is a `t_VECSMALL` of primes and E a `t_VECSMALL` of exponents.

`ulong eulerphiu(ulong n)`, Euler's totient function of n.

`ulong eulerphiu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long moebiusu(ulong n)`, Moebius μ-function of n.

`long moebiusu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong radicalu(ulong n)`, product of primes dividing n.

`GEN divisorsu(ulong n)`, returns the divisors of n in a `t_VECSMALL`, sorted by increasing order.

`GEN divisorsu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`GEN divisors_fact_degmax(GEN fa, long degmax)` return the list of divisors of degree at most `degmax` of the polynomial whose factorization is given by `fa`.

`GEN divisorsu_fact_factored(GEN fa)` where `fa` is `factoru(n)`. Return a vector $[D, F]$, where D is a a `t_VECSMALL` containing the divisors of u and $F[i]$ contains `factoru`$(D[i])$.

`GEN divisorsu_moebius(GEN P)` returns the divisors of n of the form $\prod_{p\in S}(-p)$, $S \subset P$ in a `t_VECSMALL`. The vector is not sorted but its first element is guaranteed to be 1. If P is `factoru(n)[1]`, this returns the set of $\mu(d)d$ where d runs through the squarefree divisors of n.

`long numdivu(ulong n)`, returns the number of positive divisors of $n > 0$.

`long numdivu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long omegau(ulong n)` returns the number of prime divisors of $n > 0$.

`long maxomegau(ulong x)` return the optimal B such that $\omega(n) \leq B$ for all $n \leq x$.

`long maxomegaoddu(ulong x)` return the optimal B such that $\omega(n) \leq B$ for all odd $n \leq x$.

`long uissquarefree(ulong n)` returns 1 if `n` is square-free, and 0 otherwise.

`long uissquarefree_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long uposisfundamental(ulong x)` return 1 if x is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 if $-x$ is a fundamental discriminant, and 0 otherwise.

`long sisfundamental(long x)` return 1 if x is a fundamental discriminant, and 0 otherwise.

`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong` x.

`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$.

`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong` n.

`int uisnilpotent(ulong n)` return 1 if every finite group or order n is nilpotent, 0 otherwise.

`int uislucaspsp(ulong n)` returns 1 if the `ulong` n fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`int uis2psp(ulong n)` returns 1 if the odd `ulong` $n > 1$ fails a strong Rabin-Miller test for the base 2 (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`int uispsp(ulong a, ulong n)` returns 1 if the odd `ulong` $n > 1$ fails a strong Rabin-Miller test for the base $1 < a < n$ (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`ulong sumdigitsu(ulong n)` returns the sum of decimal digits of u.

`GEN usumdiv_fact(GEN fa)`, sum of divisors of `ulong` n, where `fa` is `factoru(n)`.

`GEN usumdivk_fact(GEN fa, ulong k)`, sum of k-th powers of divisors of `ulong` n, where `fa` is `factoru(n)`.

`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol (x, y) at the prime p (`NULL` for the place at infinity); x and y are `t_INT`s.

`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum attached to the `t_INT` h and k, $k > 0$.

`GEN sumdedekind_coprime(GEN h, GEN k)` as `sumdedekind`, except that h and k are assumed to be coprime `t_INT`s.

`GEN u_sumdedekind_coprime(long h, long k)` Let $k > 0$, $0 \leq h < k$, $(h, k) = 1$. Returns $[s_1, s_2]$ in a `t_VECSMALL`, such that $s(h, k) = (s_2 + ks_1)/(12k)$. Requires $\max(h + k/2, k) <$ `LONG_MAX` to avoid overflow, in particular $k \leq (2/3)$`LONG_MAX` is fine.

Chapter 7:
Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

7.1 Naming scheme.

A function name is built in the following way: $A_1\_\ldots\_A_n fun$ for an operation *fun* with n arguments of class $A_1,\ldots,A_n$. A class name is given by a base ring followed by a number of code letters. Base rings are among

`Fl`: $\mathbf{Z}/l\mathbf{Z}$ where $l < 2^{\texttt{BITS\_IN\_LONG}}$ is not necessarily prime. Implemented using `ulong`s

`Fp`: $\mathbf{Z}/p\mathbf{Z}$ where p is a `t_INT`, not necessarily prime. Implemented as `t_INT`s z, preferably satisfying $0 \le z < p$. More precisely, any `t_INT` can be used as an `Fp`, but reduced inputs are treated more efficiently. Outputs from `Fp`xxx routines are reduced.

`Fq`: $\mathbf{Z}[X]/(p, T(X))$, p a `t_INT`, T a `t_POL` with `Fp` coefficients or `NULL` (in which case no reduction modulo `T` is performed). Implemented as `t_POL`s z with `Fp` coefficients, $\deg(z) < \deg \texttt{T}$, although z a `t_INT` is allowed for elements in the prime field.

`Z`: the integers $\mathbf{Z}$, implemented as `t_INT`s.

`Zp`: the p-adic integers $\mathbf{Z}_p$, implemented as `t_INT`s, for arbitrary p

`Zl`: the p-adic integers $\mathbf{Z}_p$, implemented as `t_INT`s, for $p < 2^{\texttt{BITS\_IN\_LONG}}$

`z`: the integers $\mathbf{Z}$, implemented using (signed) `long`s.

`Q`: the rational numbers $\mathbf{Q}$, implemented as `t_INT`s and `t_FRAC`s.

`Rg`: a commutative ring, whose elements can be `gadd`-ed, `gmul`-ed, etc.

Possible letters are:

`X`: polynomial in X (`t_POL` in a fixed variable), e.g. `FpX` means $\mathbf{Z}/p\mathbf{Z}[X]$

`Y`: polynomial in $Y \neq X$. This is used to resolve ambiguities. E.g. `FpXY` means $((\mathbf{Z}/p\mathbf{Z})[X])[Y]$.

`V`: vector (`t_VEC` or `t_COL`), treated as a row vector (independently of the actual type). E.g. `ZV` means $\mathbf{Z}^k$ for some k.

`C`: vector (`t_VEC` or `t_COL`), treated as a column vector (independently of the actual type). The difference with `V` is purely semantic: if the result is a vector, it will be of type `t_COL` unless mentioned otherwise. For instance the function `ZC_add` receives two integral vectors (`t_COL` or `t_VEC`, possibly different types) of the same length and returns a `t_COL` whose entries are the sums of the input coefficients.

`M`: matrix (`t_MAT`). E.g. `QM` means a matrix with rational entries

`T`: Trees. Either a leaf or a `t_VEC` of trees.

`E`: point over an elliptic curve, represented as two-component vectors `[x,y]`, except for the represented by the one-component vector `[0]`. Not all curve models are supported.

`Q`: representative (`t_POL`) of a class in a polynomial quotient ring. E.g. an `FpXQ` belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$, `FpXQV` means a vector of such elements, etc.

`n`: a polynomial representative (`t_POL`) for a truncated power series modulo X^n. E.g. an `FpXn` belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, `FpXnV` means a vector of such elements, etc.

`x`, `y`, `m`, `v`, `c`, `q`: as their uppercase counterpart, but coefficient arrays are implemented using `t_VECSMALL`s, which coefficient understood as `ulong`s.

`x` and `y` (and `q`) are implemented by a `t_VECSMALL` whose first coefficient is used as a code-word and the following are the coefficients , similarly to a `t_POL`. This is known as a 'POLSMALL'.

`m` are implemented by a `t_MAT` whose components (columns) are `t_VECSMALL`s. This is known as a 'MATSMALL'.

`v` and `c` are regular `t_VECSMALL`s. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

`add`: add

`sub`: subtract

`mul`: multiply

`sqr`: square

`div`: divide (Euclidean quotient)

`rem`: Euclidean remainder

`divrem`: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: `NULL` (do not store the remainder, used to implement `div`), `ONLY_REM` (return the remainder, used to implement `rem`), `ONLY_DIVIDES` (return the quotient if the division is exact, and `NULL` otherwise).

`gcd`: GCD

`extgcd`: return GCD, store Bezout coefficients in pointer arguments

`pow`: exponentiate

`eval`: evaluation / composition

7.2 Coefficient ring.

`long Rg_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the object x is defined.

Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g. $\mathbf{F}_p$ and $\mathbf{F}_q$ for primes $p \neq q$, $\mathbf{F}_p[X]/(T)$ and $\mathbf{F}_p[X]/(U)$ for $T \neq U$). Minor discrepancies are supported if they make general sense (e.g. $\mathbf{F}_p$ and $\mathbf{F}_{p^k}$, but not $\mathbf{F}_p$ and $\mathbf{Q}_p$); `t_FFELT` and `t_POLMOD` of `t_INTMOD`s are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

• 0: none of the others (presumably multivariate, possibly inconsistent).

• `t_INT`: defined over $\mathbf{Z}$.

• `t_FRAC`: defined over $\mathbf{Q}$.

• `t_INTMOD`: defined over $\mathbf{Z}/p\mathbf{Z}$, where `*ptp` is set to p. It is not checked whether p is prime.

• `t_COMPLEX`: defined over $\mathbf{C}$ (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.

• `t_REAL`: defined over $\mathbf{R}$ (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.

• `t_PADIC`: defined over $\mathbf{Q}_p$, where `*ptp` is set to p and `*ptprec` to the p-adic accuracy.

• `t_FFELT`: defined over a finite field $\mathbf{F}_{p^k}$, where `*ptp` is set to the field characteristic p and `*ptpol` is set to a `t_FFELT` belonging to the field.

• `t_POL`: defined over a polynomial ring.

• other values are composite corresponding to quotients $R[X]/(T)$, with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing R. If `t` is the `RgX_type`, `t1` and `t2` are recovered using

`void RgX_type_decode(long t, long *t1, long *t2)`

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to $y^2 + 1$.

and the underlying base ring R is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`long RgX_type_code(long t1, long t2)` is the reverse of `RgX_type_decode`, it returns the compound type t built from `t1` and `t2`. This is implemented as a macro so as to be usable as a label in a switch when both types are constants, e.g.,

```
case RgX_type_code(t_POLMOD, t_INTMOD): ...
```

`int RgX_type_is_composite(long t)` t as returned by `RgX_type`, return 1 if t is a composite type, and 0 otherwise.

`GEN Rg_get_0(GEN x)` returns 0 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`, `O(5^10)`).

`GEN Rg_get_1(GEN x)` returns 1 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`,

`long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the polynomial x is defined, otherwise as `Rg_type`.

`long RgX_Rg_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the polynomial x and the element y are defined, otherwise as `Rg_type`.

`long RgX_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the polynomials x and y are defined, otherwise as `Rg_type`.

`long RgX_type3(GEN x, GEN y, GNE z, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the polynomials x, y and z are defined, otherwise as `Rg_type`.

`long RgM_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the matrix x is defined, otherwise as `Rg_type`.

`long RgM_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the matrices x and y are defined, otherwise as `Rg_type`.

`long RgV_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the vector x is defined, otherwise as `Rg_type`.

`long RgV_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the vectors x and y are defined, otherwise as `Rg_type`.

`long RgM_RgC_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the matrix x and the vector y are defined, otherwise as `Rg_type`.

7.3 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, where p is not necessarily prime and $T \in (\mathbf{Z}/p\mathbf{Z})[X]$ is possibly reducible; and finite extensions thereof. All this can be emulated with `t_INTMOD` and `t_POLMOD` coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

7.3.1 `FpC / FpV, FpM`. A `ZV` (resp. a `ZM`) is a `t_VEC` or `t_COL` (resp. `t_MAT`) with `t_INT` coefficients. An `FpV` or `FpM`, with respect to a given `t_INT p`, is the same with `Fp` coordinates; operations are understood over $\mathbf{Z}/p\mathbf{Z}$.

7.3.1.1 Conversions.

`int Rg_is_Fp(GEN z, GEN *p)`, checks if `z` can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT` or a `t_INTMOD` whose modulus is equal to `*p`, (if `*p` not `NULL`), in that case return 1, else 0. If a modulus is found it is put in `*p`, else `*p` is left unchanged.

`int RgV_is_FpV(GEN z, GEN *p)`, `z` a `t_VEC` (resp. `t_COL`), checks if it can be mapped to a `FpV` (resp. `FpC`), by checking `Rg_is_Fp` coefficientwise.

`int RgM_is_FpM(GEN z, GEN *p)`, `z` a `t_MAT`, checks if it can be mapped to a `FpM`, by checking `RgV_is_FpV` columnwise.

`GEN Rg_to_Fp(GEN z, GEN p)`, `z` a scalar which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT`, a `t_INTMOD` whose modulus is divisible by p, a `t_FRAC` whose denominator is coprime to p, or a `t_PADIC` with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized.

`GEN padic_to_Fp(GEN x, GEN p)` special case of `Rg_to_Fp`, for a x a `t_PADIC`.

`GEN RgV_to_FpV(GEN z, GEN p)`, `z` a `t_VEC` or `t_COL`, returns the `FpV` (as a `t_VEC`) obtained by applying `Rg_to_Fp` coefficientwise.

`GEN RgC_to_FpC(GEN z, GEN p)`, `z` a `t_VEC` or `t_COL`, returns the `FpC` (as a `t_COL`) obtained by applying `Rg_to_Fp` coefficientwise.

`GEN RgM_to_FpM(GEN z, GEN p)`, `z` a `t_MAT`, returns the `FpM` obtained by applying `RgC_to_FpC` columnwise.

`GEN RgM_Fp_init(GEN z, GEN p, ulong *pp)`, given an `RgM` z, whose entries can be mapped to $\mathbf{F}_p$ (as per `Rg_to_Fp`), and a prime number p. This routine returns a normal form of z: either an `F2m` ($p = 2$), an `Flm` (p fits into an `ulong`) or an `FpM`. In the first two cases, `pp` is set to `itou`(p), and to 0 in the last.

The functions above are generally used as follows:

```
GEN add(GEN x, GEN y)
{
  GEN p = NULL;
  if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
  {
    x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
    z = Fp_add(x, y, p);
    return Fp_to_mod(z);
  }
  else return gadd(x, y);
}
```

`GEN FpC_red(GEN z, GEN p)`, `z` a `ZC`. Returns `lift(Col(z) * Mod(1,p))`, hence a `t_COL`.

`GEN FpV_red(GEN z, GEN p)`, `z` a `ZV`. Returns `lift(Vec(z) * Mod(1,p))`, hence a `t_VEC`

`GEN FpM_red(GEN z, GEN p)`, `z` a `ZM`. Returns `lift(z * Mod(1,p))`, which is an `FpM`.

7.3.1.2 Basic operations.

`GEN random_FpC(long n, GEN p)` returns a random `FpC` with n components.

`GEN random_FpV(long n, GEN p)` returns a random `FpV` with n components.

`GEN FpC_center(GEN z, GEN p, GEN pov2)` returns a `t_COL` whose entries are the `Fp_center` of the `gel(z,i)`.

`GEN FpM_center(GEN z, GEN p, GEN pov2)` returns a matrix whose entries are the `Fp_center` of the `gcoeff(z,i,j)`.

`void FpC_center_inplace(GEN z, GEN p, GEN pov2)` in-place version of `FpC_center`, using `affii`.

`void FpM_center_inplace(GEN z, GEN p, GEN pov2)` in-place version of `FpM_center`, using `affii`.

`GEN FpC_add(GEN x, GEN y, GEN p)` adds the `ZC` x and y and reduce modulo p to obtain an `FpC`.

`GEN FpV_add(GEN x, GEN y, GEN p)` same as `FpC_add`, returning and `FpV`.

`GEN FpM_add(GEN x, GEN y, GEN p)` adds the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_sub(GEN x, GEN y, GEN p)` subtracts the `ZC` y to the `ZC` x and reduce modulo p to obtain an `FpC`.

`GEN FpV_sub(GEN x, GEN y, GEN p)` same as `FpC_sub`, returning and `FpV`.

`GEN FpM_sub(GEN x, GEN y, GEN p)` subtracts the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `ZC` `x` (seen as a column vector) by the `t_INT` `y` and reduce modulo `p` to obtain an `FpC`.

`GEN FpM_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `ZM` `x` (seen as a column vector) by the `t_INT` `y` and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_FpV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC` `x` (seen as a column vector) by the `ZV` `y` (seen as a row vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_mul(GEN x, GEN y, GEN p)` multiplies the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_powu(GEN x, ulong n, GEN p)` computes x^n where x is a square `FpM`.

`GEN FpM_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZM` `x` by the `ZC` `y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpC`.

`GEN FpM_FpC_mul_FpX(GEN x, GEN y, GEN p, long v)` is a stack-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

`GEN FpV_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZV` `x` (seen as a row vector) by the `ZC` `y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `Fp`.

GEN FpV_dotproduct(GEN x, GEN y, GEN p) scalar product of x and y (assumed to have the same length).

GEN FpV_dotsquare(GEN x, GEN p) scalar product of x with itself. has t_INT entries.

GEN FpV_factorback(GEN L, GEN e, GEN p) given an FpV L and a ZV or zv e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

7.3.1.3 Fp-linear algebra. The implementations are not asymptotically efficient ($O(n^3)$ standard algorithms). Unless mentioned otherwise, all these functions assume that p is prime.

GEN FpM_deplin(GEN x, GEN p) returns a nontrivial kernel vector, or NULL if none exist.

GEN FpM_det(GEN x, GEN p) as det

GEN FpM_gauss(GEN a, GEN b, GEN p) as gauss, where a and b are FpM.

GEN FpM_FpC_gauss(GEN a, GEN b, GEN p) as gauss, where a is a FpM and b a FpC.

GEN FpM_image(GEN x, GEN p) as image

GEN FpM_intersect(GEN x, GEN y, GEN p) as intersect

GEN FpM_intersect_i(GEN x, GEN y, GEN p) internal variant of FpM_intersect but the result is only a generating set, not necessarily an $\mathbf{F}_p$-basis. It is not stack-clean, but suitable for gc_upto.

GEN FpM_inv(GEN x, GEN p) returns a left inverse of x (the inverse if x is square), or NULL if x is not invertible.

GEN FpM_FpC_invimage(GEN A, GEN y, GEN p) given an FpM A and an FpC y, returns an x such that $Ax = y$, or NULL if no such vector exist.

GEN FpM_invimage(GEN A, GEN y, GEN p) given two FpM A and y, returns x such that $Ax = y$, or NULL if no such matrix exist.

GEN FpM_ker(GEN x, GEN p) as ker

long FpM_rank(GEN x, GEN p) as rank

GEN FpM_indexrank(GEN x, GEN p) as indexrank

GEN FpM_suppl(GEN x, GEN p) as suppl

GEN FpM_hess(GEN x, GEN p) upper Hessenberg form of x over $\mathbf{F}_p$.

GEN FpM_charpoly(GEN x, GEN p) characteristic polynomial of x.

GEN FpM_trace(GEN x, GEN p) trace of x. The integer p need not be prime.

7.3.1.4 FqC, FqM and Fq-linear algebra.

An FqM (resp. FqC) is a matrix (resp a t_COL) with Fq coefficients (with respect to given T, p), not necessarily reduced (i.e arbitrary t_INTs and ZXs in the same variable as T).

GEN RgC_to_FqC(GEN z, GEN T, GEN p)

GEN RgM_to_FqM(GEN z, GEN T, GEN p)

GEN FqC_to_FlxqC(GEN x, GEN T, ulong p) where T is a non-NULL Flx.

GEN FqM_to_FlxqM(GEN x, GEN T, ulong p) where T is a non-NULL Flx.

GEN FqC_add(GEN a, GEN b, GEN T, GEN p)

GEN FqC_sub(GEN a, GEN b, GEN T, GEN p)

GEN FqC_Fq_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqC_FqV_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM_add(GEN a, GEN b, GEN T, GEN p)

GEN FqM_sub(GEN a, GEN b, GEN T, GEN p)

GEN FqM_FqC_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM_FqC_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where b is a FqC.

GEN FqM_FqC_invimage(GEN a, GEN b, GEN T, GEN p)

GEN FqM_deplin(GEN x, GEN T, GEN p) returns a nontrivial kernel vector, or NULL if none exist.

GEN FqM_det(GEN x, GEN T, GEN p) as det

GEN FqM_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where b is a FqM.

GEN FqM_image(GEN x, GEN T, GEN p) as image

GEN FqM_indexrank(GEN x, GEN T, GEN p) as indexrank

GEN FqM_inv(GEN x, GEN T, GEN p) returns the inverse of x, or NULL if x is not invertible.

GEN FqM_invimage(GEN a, GEN b, GEN T, GEN p) as invimage

GEN FqM_ker(GEN x, GEN T, GEN p) as ker

GEN FqM_mul(GEN a, GEN b, GEN T, GEN p)

long FqM_rank(GEN x, GEN T, GEN p) as rank

GEN FqM_suppl(GEN x, GEN T, GEN p) as suppl

7.3.2 Flc / Flv, Flm. See FpV, FpM operations.

GEN Flv_copy(GEN x) returns a copy of x.

GEN Flv_center(GEN z, ulong p, ulong ps2)

GEN random_Flv(long n, ulong p) returns a random Flv with n components.

GEN Flm_copy(GEN x) returns a copy of x.

GEN matid_Flm(long n) returns an Flm which is an $n \times n$ identity matrix.

GEN scalar_Flm(long s, long n) returns an Flm which is s times the $n \times n$ identity matrix.

GEN Flm_center(GEN z, ulong p, ulong ps2)

GEN Flm_Fl_add(GEN x, ulong y, ulong p) returns $x + y * \mathrm{Id}$ (x must be square).

GEN Flm_Fl_sub(GEN x, ulong y, ulong p) returns $x - y * \mathrm{Id}$ (x must be square).

GEN Flm_Flc_mul(GEN x, GEN y, ulong p) multiplies x and y (assumed to have compatible dimensions).

GEN Flm_Flc_mul_pre(GEN x, GEN y, ulong p, ulong pi) multiplies x and y (assumed to have compatible dimensions), assuming pi is the pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

`GEN Flc_Flv_mul(GEN x, GEN y, ulong p)` multiplies the column vector x by the row vector y. The result is a matrix.

`GEN Flm_Flc_mul_pre_Flx(GEN x, GEN y, ulong p, ulong pi, long sv)` return `Flv_to_Flx(Flm_Flc_mul_pre(x, y, p, pi), sv)`.

`GEN Flm_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flm x` by `y`.

`GEN Flm_Fl_mul_pre(GEN x, ulong y, ulong p, ulong pi)` multiplies the `Flm x` by `y` assuming pi is the pseudoinverse of p, or 0 in which case we assume $p < B^{1/2}$ is small.

`GEN Flm_neg(GEN x, ulong p)` negates the `Flm x`.

`void Flm_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flm x` by `x * y`.

`GEN Flv_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flv x` by `y`.

`void Flv_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flc x` by `x * y`.

`void Flv_Fl_mul_part_inplace(GEN x, ulong y, ulong p, long l)` multiplies $x[1..l]$ by y modulo p. In place.

`GEN Flv_Fl_div(GEN x, ulong y, ulong p)` divides the `Flv x` by `y`.

`void Flv_Fl_div_inplace(GEN x, ulong y, ulong p)` replaces the `Flv x` by `x/y`.

`void Flc_lincomb1_inplace(GEN X, GEN Y, ulong v, ulong q)` sets $X \leftarrow X + vY$, where X, Y are `Flc`. Memory efficient (e.g. no-op if $v = 0$); if the input was gc-safe, it remains so.

`GEN Flv_add(GEN x, GEN y, ulong p)` adds two `Flv`.

`void Flv_add_inplace(GEN x, GEN y, ulong p)` replaces x by $x + y$.

`GEN Flv_neg(GEN x, ulong p)` returns $-x$.

`void Flv_neg_inplace(GEN x, ulong p)` replaces x by $-x$.

`GEN Flv_sub(GEN x, GEN y, ulong p)` subtracts `y` to `x`.

`void Flv_sub_inplace(GEN x, GEN y, ulong p)` replaces x by $x - y$.

`ulong Flv_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of `x` and `y`

`ulong Flv_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)` returns the scalar product of `x` and `y` assuming pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flv_factorback(GEN L, GEN e, ulong p)` given an `Flv` L and a `zv` e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

`ulong Flv_sum(GEN x, ulong p)` returns the sum of the components of x.

`ulong Flv_prod(GEN x, ulong p)` returns the product of the components of x.

`ulong Flv_prod_pre(GEN x, ulong p, ulong pi)` as `Flv_prod` assuming pi is the pseudoinverse of p.

`GEN Flv_inv(GEN x, ulong p)` returns the vector of inverses of the elements of x (as a `Flv`). Use Montgomery's trick.

`void Flv_inv_inplace(GEN x, ulong p)` in place variant of `Flv_inv`.

`GEN Flv_inv_pre(GEN x, ulong p, ulong pi)` as `Flv_inv` assuming pi is the pseudoinverse of p.

`void Flv_inv_pre_inplace(GEN x, ulong p, ulong pi)` in place variant of `Flv_inv`.

`GEN Flc_FpV_mul(GEN x, GEN y, GEN p)` multiplies x (seen as a column vector) by y (seen as a row vector, assumed to have compatible dimensions) to obtain an `Flm`.

`GEN zero_Flm(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_Flm_copy(long m, long n)` creates a `Flm` with `m` x `n` components set to 0.

`GEN zero_Flv(long n)` creates a `Flv` with `n` components set to 0.

`GEN Flm_row(GEN A, long x0)` return $A[i,]$, the i-th row of the `Flm` A.

`GEN Flm_add(GEN x, GEN y, ulong p)` adds `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_sub(GEN x, GEN y, ulong p)` subtracts `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_mul_pre(GEN x, GEN y, ulong p, ulong pi)` multiplies `x` and `y` (assumed to have compatible dimensions), assuming pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flm_sqr(GEN x, ulong p)` squares `x` (assumed to be square).

`GEN Flm_powers(GEN x, ulong n, ulong p)` returns $[\mathtt{x}^0, \dots, \mathtt{x}^\mathtt{n}]$ as a `t_VEC` of `Flm`s.

`GEN Flm_powu(GEN x, ulong n, ulong p)` computes x^n where x is a square `Flm`.

`GEN Flm_charpoly(GEN x, ulong p)` return the characteristic polynomial of the square `Flm` x, as a `Flx`.

`GEN Flm_deplin(GEN x, ulong p)` Assume that `p` is prime.

`ulong Flm_det(GEN x, ulong p)` Assume that `p` is prime.

`ulong Flm_det_sp(GEN x, ulong p)`, as `Flm_det`, in place (destroys `x`). Assume that `p` is prime.

`ulong Flm_trace(GEN x, ulong p)`

`GEN Flm_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flm`. Assume that `p` is prime.

`GEN Flm_Flc_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flc`. Assume that `p` is prime.

`GEN Flm_indexrank(GEN x, ulong p)` Assume that `p` is prime.

`GEN Flm_inv(GEN x, ulong p)` Assume that `p` is prime.

`GEN Flm_adjoint(GEN x, ulong p)` as `matadjoint`. Assume that `p` is prime.

`GEN Flm_Flc_invimage(GEN A, GEN y, ulong p)` given an `Flm` A and an `Flc` y, returns an x such that $Ax = y$, or `NULL` if no such vector exist. Assume that `p` is prime.

`GEN Flm_invimage(GEN A, GEN y, ulong p)` given two `Flm` A and y, returns x such that $Ax = y$, or `NULL` if no such matrix exist. Assume that `p` is prime.

`GEN Flm_ker(GEN x, ulong p)` Assume that `p` is prime.

`GEN Flm_ker_sp(GEN x, ulong p, long deplin)`, as `Flm_ker` (if `deplin=0`) or `Flm_deplin` (if `deplin=1`) , in place (destroys `x`). Assume that `p` is prime.

`long Flm_rank(GEN x, ulong p)` Assume that `p` is prime.

`long Flm_suppl(GEN x, ulong p)` Assume that `p` is prime.

`GEN Flm_image(GEN x, ulong p)` Assume that `p` is prime.

`GEN Flm_intersect(GEN x, GEN y, ulong p)` Assume that `p` is prime.

`GEN Flm_intersect_i(GEN x, GEN y, GEN p)` internal variant of `Flm_intersect` but the result is only a generating set, not necessarily an $\mathbf{F}_p$-basis. It *is* a basis if both x and y have independent columns. The function is not stack-clean, but suitable for `gc_upto`. Assume that `p` is prime.

`GEN Flm_transpose(GEN x)`

`GEN Flm_hess(GEN x, ulong p)` upper Hessenberg form of x over $\mathbf{F}_p$. Assume that `p` is prime.

`GEN Flm_hess_pre(GEN x, ulong p, ulong pi)` as `Flm_hess` assuming pi is the pseudoinverse of p.

7.3.3 `F2c / F2v, F2m`. An `F2v v` is a `t_VECSMALL` representing a vector over $\mathbf{F}_2$. Specifically `z[0]` is the usual codeword, `z[1]` is the number of components of v and the coefficients are given by the bits of remaining words by increasing indices.

`ulong F2v_coeff(GEN x, long i)` returns the coefficient $i \geq 1$ of x.

`void F2v_clear(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 0.

`int F2v_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

`void F2v_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 1$ of x.

`void F2v_set(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 1.

`void F2v_copy(GEN x)` returns a copy of x.

`GEN F2v_slice(GEN x, long a, long b)` returns the `F2v` with entries $x[a], \ldots, x[b]$. Assumes $a \leq b$.

`ulong F2m_coeff(GEN x, long i, long j)` returns the coefficient (i, j) of x.

`void F2m_clear(GEN x, long i, long j)` sets the coefficient (i, j) of x to 0.

`void F2m_flip(GEN x, long i, long j)` adds 1 to the coefficient (i, j) of x.

`void F2m_set(GEN x, long i, long j)` sets the coefficient (i, j) of x to 1.

`GEN F2m_copy(GEN x)` returns a copy of x.

`GEN F2m_transpose(GEN x)` returns the transpose of x.

`GEN F2m_row(GEN x, long j)` returns the `F2v` which corresponds to the j-th row of the `F2m` x.

`GEN F2m_rowslice(GEN x, long a, long b)` returns the `F2m` built from the a-th to b-th rows of the `F2m` x. Assumes $a \leq b$.

`GEN F2m_F2c_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_image(GEN x)` gives a subset of the columns of x that generate the image of x.

`GEN F2m_invimage(GEN A, GEN B)`

`GEN F2m_F2c_invimage(GEN A, GEN y)`

`GEN F2m_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2m`.

`GEN F2m_F2c_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2c`.

`GEN F2m_indexrank(GEN x)` x being a matrix of rank r, returns a vector with two `t_VECSMALL` components y and z of length r giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract`(x, y, z) is invertible.

`GEN F2m_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_powu(GEN x, ulong n)` computes x^n where x is a square `F2m`.

`long F2m_rank(GEN x)` as `rank`.

`long F2m_suppl(GEN x)` as `suppl`.

`GEN matid_F2m(long n)` returns an `F2m` which is an $n \times n$ identity matrix.

`GEN zero_F2v(long n)` creates a `F2v` with `n` components set to 0.

`GEN const_F2v(long n)` creates a `F2v` with `n` components set to 1.

`GEN F2v_ei(long n, long i)` creates a `F2v` with `n` components set to 0, but for the i-th one, which is set to 1 (i-th vector in the canonical basis).

`GEN zero_F2m(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_F2m_copy(long m, long n)` creates a `F2m` with `m` x `n` components set to 0.

`GEN F2v_to_Flv(GEN x)`

`GEN F2c_to_ZC(GEN x)`

`GEN ZV_to_F2v(GEN x)`

`GEN RgV_to_F2v(GEN x)`

`GEN F2m_to_Flm(GEN x)`

`GEN F2m_to_ZM(GEN x)`

`GEN Flv_to_F2v(GEN x)`

`GEN Flm_to_F2m(GEN x)`

`GEN ZM_to_F2m(GEN x)`

`GEN RgM_to_F2m(GEN x)`

`void F2v_add_inplace(GEN x, GEN y)` replaces x by $x + y$. It is allowed for y to be shorter than x.

`void F2v_and_inplace(GEN x, GEN y)` replaces x by the term-by term product of x and y (which is the logical `and`). It is allowed for y to be shorter than x.

`void F2v_negimply_inplace(GEN x, GEN y)` replaces x by the term-by term logical `and not` of x and y. It is allowed for y to be shorter than x.

`void F2v_or_inplace(GEN x, GEN y)` replaces x by the term-by term logical `or` of x and y. It is allowed for y to be shorter than x.

`int F2v_subset(GEN x, GEN y)` return 1 if the set of indices of non-zero components of y is a subset of the set of indices of non-zero components of x, 0 otherwise.

`ulong F2v_hamming(GEN x)` returns the Hamming weight of `x`, that is the number of nonzero entries.

`ulong F2m_det(GEN x)`

`ulong F2m_det_sp(GEN x)`, as `F2m_det`, in place (destroys `x`).

`GEN F2m_deplin(GEN x)`

`ulong F2v_dotproduct(GEN x, GEN y)` returns the scalar product of `x` and `y`

`GEN F2m_inv(GEN x)`

`GEN F2m_ker(GEN x)`

`GEN F2m_ker_sp(GEN x, long deplin)`, as `F2m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys `x`).

7.3.4 `F3c / F3v, F3m`. An `F3v v` is a `t_VECSMALL` representing a vector over $\mathbf{F}_3$. Specifically `z[0]` is the usual codeword, `z[1]` is the number of components of v and the coefficients are given by pair of adjacent bits of remaining words by increasing indices, with the coding $00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 2$ and 11 is undefined.

`ulong F3v_coeff(GEN x, long i)` returns the coefficient $i \geq 1$ of x.

`void F3v_clear(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 0.

`void F3v_set(GEN x, long i, ulong n)` sets the coefficient $i \geq 1$ of x to $n < 3$,

`ulong F3m_coeff(GEN x, long i, long j)` returns the coefficient (i, j) of x.

`void F3m_set(GEN x, long i, long j, ulong n)` sets the coefficient (i, j) of x to $n < 3$.

`GEN F3m_copy(GEN x)` returns a copy of x.

`GEN F3m_transpose(GEN x)` returns the transpose of x.

`GEN F3m_row(GEN x, long j)` returns the `F3v` which corresponds to the j-th row of the `F3m` x.

`GEN F3m_ker(GEN x)`

`GEN F3m_ker_sp(GEN x, long deplin)`, as `F3m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys `x`).

`GEN F3m_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`long F3m_rank(GEN x)`

`GEN F3v_dotproduct(GEN x, GEN y)`

`GEN F3v_dotsquare(GEN x)`

`GEN zero_F3v(long n)` creates a `F3v` with `n` components set to 0.

`GEN zero_F3m_copy(long m, long n)` creates a `F3m` with `m` x `n` components set to 0.

GEN F3v_to_Flv(GEN x)

GEN ZV_to_F3v(GEN x)

GEN RgV_to_F3v(GEN x)

GEN F3c_to_ZC(GEN x)

GEN F3m_to_Flm(GEN x)

GEN F3m_to_ZM(GEN x)

GEN Flv_to_F3v(GEN x)

GEN Flm_to_F3m(GEN x)

GEN ZM_to_F3m(GEN x)

GEN RgM_to_F3m(GEN x)

7.3.5 FlxqV, FlxqC, FlxqM. See FqV, FqC, FqM operations.

GEN FlxqV_dotproduct(GEN x, GEN y, GEN T, ulong p) as FpV_dotproduct.

GEN FlxqV_dotproduct_pre(GEN x, GEN y, GEN T, ulong p, ulong pi) where pi is the pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN FlxM_Flx_add_shallow(GEN x, GEN y, ulong p) as RgM_Rg_add_shallow.

GEN FlxqC_Flxq_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM_Flxq_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM_FlxqC_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_FlxqC_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_FlxqC_mul(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_deplin(GEN x, GEN T, ulong p)

GEN FlxqM_det(GEN x, GEN T, ulong p)

GEN FlxqM_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_image(GEN x, GEN T, ulong p)

GEN FlxqM_indexrank(GEN x, GEN T, ulong p)

GEN FlxqM_inv(GEN x, GEN T, ulong p)

GEN FlxqM_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_ker(GEN x, GEN T, ulong p)

GEN FlxqM_mul(GEN a, GEN b, GEN T, ulong p)

long FlxqM_rank(GEN x, GEN T, ulong p)

GEN FlxqC_red_pre(GEN V, ulong p, ulong pi)

GEN FlxqM_red_pre(GEN M, ulong p, ulong pi)

GEN FlxqM_suppl(GEN x, GEN T, ulong p)

GEN matid_FlxqM(long n, GEN T, ulong p)

7.3.6 FpX. Let p an understood t_INT, to be given in the function arguments; in practice p is not assumed to be prime, but be wary. Recall than an Fp object is a t_INT, preferably belonging to $[0, \mathtt{p}-1]$; an FpX is a t_POL in a fixed variable whose coefficients are Fp objects. Unless mentioned otherwise, all outputs in this section are FpXs. All operations are understood to take place in $(\mathbf{Z}/\mathtt{p}\mathbf{Z})[X]$.

7.3.6.1 Conversions. In what follows p is always a t_INT, not necessarily prime.

int RgX_is_FpX(GEN z, GEN *p), z a t_POL, checks if it can be mapped to a FpX, by checking Rg_is_Fp coefficientwise.

GEN RgX_to_FpX(GEN z, GEN p), z a t_POL, returns the FpX obtained by applying Rg_to_Fp coefficientwise.

GEN FpX_red(GEN z, GEN p), z a ZX, returns lift(z * Mod(1,p)), normalized.

GEN FpXV_red(GEN z, GEN p), z a t_VEC of ZX. Applies FpX_red componentwise and returns the result (and we obtain a vector of FpXs).

GEN FpXT_red(GEN z, GEN p), z a tree of ZX. Applies FpX_red to each leaf and returns the result (and we obtain a tree of FpXs).

7.3.6.2 Basic operations. In what follows p is always a t_INT, not necessarily prime.

Now, except for p, the operands and outputs are all FpX objects. Results are undefined on other inputs.

GEN FpX_add(GEN x, GEN y, GEN p) adds x and y.

GEN FpX_neg(GEN x, GEN p) returns $-$x, the components are between 0 and p if this is the case for the components of x.

GEN FpX_renormalize(GEN x, long l), as normalizepol, where l = lg(x), in place.

GEN FpX_sub(GEN x, GEN y, GEN p) returns $x - y$.

GEN FpX_halve(GEN x, GEN p) returns z such that $2z = x$ modulo p assuming such z exists.

GEN FpX_mul(GEN x, GEN y, GEN p) returns xy.

GEN FpX_mulspec(GEN a, GEN b, GEN p, long na, long nb) see ZX_mulspec

GEN FpX_sqr(GEN x, GEN p) returns $\mathtt{x}^2$.

GEN FpX_powu(GEN x, ulong n, GEN p) returns x^n.

GEN FpX_convol(GEN x, GEN y, GEN p) return the-term by-term product of x and y.

GEN FpX_divrem(GEN x, GEN y, GEN p, GEN *pr) returns the quotient of x by y, and sets pr to the remainder.

GEN FpX_div(GEN x, GEN y, GEN p) returns the quotient of x by y.

GEN FpX_div_by_X_x(GEN A, GEN a, GEN p, GEN *r) returns the quotient of the FpX A by $(X - \mathtt{a})$, and sets r to the remainder A(a).

GEN FpX_rem(GEN x, GEN y, GEN p) returns the remainder x mod y.

long FpX_valrem(GEN x, GEN t, GEN p, GEN *r) The arguments x and e being nonzero FpX returns the highest exponent e such that $\mathtt{t}^e$ divides x. The quotient $\mathtt{x}/\mathtt{t}^e$ is returned in *r. In particular, if t is irreducible, this returns the valuation at t of x, and *r is the prime-to-t part of x.

`GEN FpX_deriv(GEN x, GEN p)` returns the derivative of `x`. This function is not stack-clean, but nevertheless suitable for `gc_upto`.

`GEN FpX_integ(GEN x, GEN p)` returns the primitive of `x` whose constant term is 0.

`GEN FpX_digits(GEN x, GEN B, GEN p)` returns a vector of `FpX` $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

`GEN FpXV_FpX_fromdigits(GEN v, GEN B, GEN p)` where $v = [c_0, \dots, c_n]$ is a vector of `FpX`, returns $\sum_{i=0}^{n} c_i B^i$.

`GEN FpX_Fp_translate(GEN P, GEN c, GEN p)` let c be an `Fp` and let P be an `FpX`; returns the translated `FpX` of $P(X + c)$.

`GEN FpX_gcd(GEN x, GEN y, GEN p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN FpX_halfgcd(GEN x, GEN y, GEN p)` returns a two-by-two `FpXM` M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

`GEN FpX_halfgcd_all(GEN x, GEN y, GEN p, GEN *pt_a, GEN *pt_b)` as `FpX_halfgcd`, in addition, if `pt_a` (resp. `pt_b`) is not `NULL`, `*pt_a` (resp. `*pt_b`) is set to `a` (resp. `b`).

`GEN FpX_extgcd(GEN x, GEN y, GEN p, GEN *u, GEN *v)` returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{u} * \mathtt{x} + *\mathtt{v} * \mathtt{y} = d$. If `u` is `NULL`, `*u` is not computed which is a bit faster. This is useful when computing the inverse of y modulo x.

`GEN FpX_center(GEN z, GEN p, GEN pov2)` returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to $]-p/2, p[$ and that `pov2` is `shifti(p,-1)`.

`GEN FpX_center_i(GEN z, GEN p, GEN pov2)` shallow internal variant of `FpX_center`.

`GEN FpX_Frobenius(GEN T, GEN p)` returns $X^p \pmod{T(X)}$.

`GEN FpX_matFrobenius(GEN T, GEN p)` returns the matrix of the Frobenius automorphism $x \mapsto x^p$ over the power basis of $\mathbf{F}_p[X]/(T)$.

7.3.6.3 Mixed operations. The following functions implement arithmetic operations between `FpX` and `Fp` operands, the result being of type `FpX`. The integer `p` need not be prime.

`GEN Z_to_FpX(GEN x, GEN p, long v)` converts a `t_INT` to a scalar polynomial in variable v, reduced modulo p.

`GEN FpX_Fp_add(GEN y, GEN x, GEN p)` add the `Fp` `x` to the `FpX` `y`.

`GEN FpX_Fp_add_shallow(GEN y, GEN x, GEN p)` add the `Fp` `x` to the `FpX` `y`. Shallow function.

`GEN FpX_Fp_sub(GEN y, GEN x, GEN p)` subtract the `Fp` `x` from the `FpX` `y`.

`GEN FpX_Fp_sub_shallow(GEN y, GEN x, GEN p)` subtract the `Fp` `x` from the `FpX` `y`. Shallow function.

`GEN Fp_FpX_sub(GEN x, GEN y, GEN p)` returns $x - y$, where x is a `t_INT` and y an `FpX`.

`GEN FpX_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `FpX` `x` by the `Fp` `y`.

`GEN FpX_Fp_mulspec(GEN x, GEN y, GEN p, long lx)` see `ZX_mulspec`

`GEN FpX_mulu(GEN x, ulong y, GEN p)` multiplies the `FpX` `x` by `y`.

`GEN FpX_Fp_mul_to_monic(GEN y, GEN x, GEN p)` returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

`GEN FpX_Fp_div(GEN x, GEN y, GEN p)` divides the `FpX x` by the `Fp y`.

`GEN FpX_divu(GEN x, ulong y, GEN p)` divides the `FpX x` by `y`.

7.3.6.4 Miscellaneous operations.

`GEN FpX_normalize(GEN z, GEN p)` divides the `FpX z` by its leading coefficient. If the latter is 1, `z` itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

`GEN FpX_invBarrett(GEN T, GEN p)`, returns the Barrett inverse M of T defined by $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T.

`GEN FpX_rescale(GEN P, GEN h, GEN p)` returns $h^{\deg(P)}P(x/h)$. `P` is an `FpX` and `h` is a nonzero `Fp` (the routine would work with any nonzero `t_INT` but is not efficient in this case). Shallow function.

`GEN FpX_eval(GEN x, GEN y, GEN p)` evaluates the `FpX x` at the `Fp y`. The result is an `Fp`.

`GEN FpX_FpV_multieval(GEN P, GEN v, GEN p)` returns the vector $[P(v[1]), \dots, P(v[n])]$ as a `FpV`.

`GEN FpX_dotproduct(GEN x, GEN y, GEN p)` return the scalar product $\sum_{i\geq 0} x_i y_i$ of the coefficients of x and y.

`GEN FpXV_FpC_mul(GEN V, GEN W, GEN p)` multiplies a nonempty row vector of `FpX` by a column vector of `Fp` of compatible dimensions. The result is an `FpX`.

`GEN FpXV_prod(GEN V, GEN p)`, `V` being a vector of `FpX`, returns their product.

`GEN FpXV_composedsum(GEN V, GEN p)`, `V` being a vector of `FpX`, returns their composed sum, see `FpX_composedsum`.

`GEN FpXV_factorback(GEN L, GEN e, GEN p, long v)` returns $\prod_i L_i^{e_i}$ where L is a vector of `FpX`s in the variable `v` and e a vector of non-negative `t_INT`s or a `t_VECSMALL`.

`GEN FpV_roots_to_pol(GEN V, GEN p, long v)`, `V` being a vector of `INT`s, returns the monic `FpX` $\prod_i (\texttt{pol\_x}[\texttt{v}] - \texttt{V}[\texttt{i}])$.

`GEN FpX_chinese_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz, GEN p)`: returns an `FpX`, congruent to `x` mod `Tx` and to `y` mod `Ty`. Assumes `Tx` and `Ty` are coprime, and `Tz = Tx * Ty` or `NULL` (in which case it is computed within).

`GEN FpV_polint(GEN x, GEN y, GEN p, long v)` returns the `FpX` interpolation polynomial with value `y[i]` at `x[i]`. Assumes lengths are the same, components are `t_INT`s, and the `x[i]` are distinct modulo `p`.

`GEN FpV_FpM_polint(GEN x, GEN V, GEN p, long v)` equivalent (but faster) to applying `FpV_polint(x,...)` to all the elements of the vector V (thus, returns a `FpXV`).

`GEN FpX_FpXV_multirem(GEN A, GEN P, GEN p)` given a `FpX` A and a vector P of pairwise coprime `FpX` of length $n \geq 1$, return a vector B of the same length such that $B[i] = A \pmod{P[i]}$ and $B[i]$ of minimal degree for all $1 \leq i \leq n$.

`GEN FpXV_chinese(GEN A, GEN P, GEN p, GEN *pM)` let P be a vector of pairwise coprime `FpX`, let A be a vector of `FpX` of the same length $n \geq 1$ and let M be the product of the elements of P.

Returns a FpX of minimal degree congruent to $A[i]$ mod $P[i]$ for all $1 \le i \le n$. If pM is not NULL, set *pM to M.

GEN FpV_invVandermonde(GEN L, GEN d, GEN p) L being a FpV of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, eventually multiplied by d if it is not NULL. If A is a FpV and $B = MA$, then the polynomial $P = \sum_{i=1}^{n} B[i]X^{i-1}$ verifies $P(L[i]) = dA[i]$ for $1 \le i \le n$.

int FpX_is_squarefree(GEN f, GEN p) returns 1 if the FpX f is squarefree, 0 otherwise.

int FpX_is_irred(GEN f, GEN p) returns 1 if the FpX f is irreducible, 0 otherwise. Assumes that p is prime. If f has few factors, FpX_nbfact(f,p) == 1 is much faster.

int FpX_is_totally_split(GEN f, GEN p) returns 1 if the FpX f splits into a product of distinct linear factors, 0 otherwise. Assumes that p is prime. The 0 polynomial is not totally split.

long FpX_ispower(GEN f, ulong k, GEN p, GEN *pt) return 1 if the FpX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN FpX_factor(GEN f, GEN p), factors the FpX f. Assumes that p is prime. The returned value v is a t_VEC with two components: v[1] is a vector of distinct irreducible (FpX) factors, and v[2] is a t_VECSMALL of corresponding exponents. The order of the factors is deterministic (the computation is not).

GEN FpX_factor_squarefree(GEN f, GEN p) returns the squarefree factorization of f modulo p. This is a vector $[u_1, \dots, u_k]$ of squarefree and pairwise coprime FpX such that $u_k \neq 1$ and $f = \prod u_i^i$. The other u_i may equal 1. Shallow function.

GEN FpX_ddf(GEN f, GEN p) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a t_VEC with two components: F=v[1] is a vector of (FpX) factors, and E=v[2] is a t_VECSMALL, such that f is equal to the product of the F[i] and each F[i] is a product of irreducible factors of degree E[i].

long FpX_ddf_degree(GEN f, GEN XP, GEN p) assuming that f is squarefree and that all its factors have the same degree, return the common degree, where XP is FpX_Frobenius(f, p).

long FpX_nbfact(GEN f, GEN p), assuming the FpX f is squarefree, returns the number of its irreducible factors. Assumes that p is prime.

long FpX_nbfact_Frobenius(GEN f, GEN XP, GEN p), as FpX_nbfact(f, p) but faster, where XP is FpX_Frobenius(f, p).

GEN FpX_degfact(GEN f, GEN p), as FpX_factor, but the degrees of the irreducible factors are returned instead of the factors themselves (as a t_VECSMALL). Assumes that p is prime.

long FpX_nbroots(GEN f, GEN p) returns the number of distinct roots in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f. Assumes that p is prime.

GEN FpX_oneroot(GEN f, GEN p) returns one root in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f. Return NULL if no root exists. Assumes that p is prime.

GEN FpX_oneroot_split(GEN f, GEN p) as FpX_oneroot. Faster when f is close to be totally split.

GEN FpX_roots(GEN f, GEN p) returns the roots in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f (without multiplicity, as a vector of Fps). Assumes that p is prime.

`GEN FpX_roots_mult(GEN f, long n, GEN p)` returns the roots in $\mathbf{Z}/p\mathbf{Z}$ with multiplicity at least n of the `FpX f` (without multiplicity, as a vector of `Fp`s). Assumes that `p` is prime.

`GEN FpX_split_part(GEN f, GEN p)` returns the largest totally split squarefree factor of f.

`GEN FpX_factcyclo(ulong n, GEN p, ulong m)` returns the factors of the n-th cyclotomic polynomial over `Fp`. if $m = 1$ returns a single factor.

`GEN random_FpX(long d, long v, GEN p)` returns a random `FpX` in variable `v`, of degree less than `d`.

`GEN FpX_resultant(GEN x, GEN y, GEN p)` returns the resultant of `x` and `y`, both `FpX`. The result is a `t_INT` belonging to $[0, p-1]$.

`GEN FpX_disc(GEN x, GEN p)` returns the discriminant of the `FpX x`. The result is a `t_INT` belonging to $[0, p-1]$.

`GEN FpX_FpXY_resultant(GEN a, GEN b, GEN p)`, `a` a `t_POL` of `t_INT`s (say in variable X), `b` a `t_POL` (say in variable X) whose coefficients are either `t_POL`s in $\mathbf{Z}[Y]$ or `t_INT`s. Returns $\mathrm{Res}_X(a, b)$ in $\mathbf{F}_p[Y]$ as an `FpY`. The function assumes that X has lower priority than Y.

`ulong FpX_extresultant(GEN a, GEN b, GEN p, GEN *ptU, GEN *ptV)` given two `FpX a` and `b`, returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set). Assumes that p is prime.

`GEN FpX_composedprod(GEN P, GEN Q, GEN p)` if $P = a\prod_{i=1}^{m}(x - p_i)$ and $Q = b\prod_{j=1}^{n}(x - q_j)$ in some suitable algebraic extension, return $a^n b^m \prod_{i,j}(x - p_i q_j)$.

`GEN FpX_composedsum(GEN P, GEN Q, GEN p)` if $P = a\prod_{i=1}^{m}(x - p_i)$ and $Q = b\prod_{j=1}^{n}(x - q_j)$ in some suitable algebraic extension, return $a^n b^m \prod_{i,j}(x - (p_i + q_j))$.

`GEN FpX_Newton(GEN x, long n, GEN p)` return $\sum i = 0^{n-1}\pi_i X^i$ where π_i is the sum of the ith-power of the roots of x in an algebraic closure.

`GEN FpX_fromNewton(GEN x, GEN p)` recover a polynomial from it s Newton sums given by the coefficients of x. This function assumes that p and the accuracy of x as a `FpXn` is larger than the degree of the solution.

`GEN FpX_Laplace(GEN x, GEN p)` return $\sum_{i=0}^{n-1} x_i i! X^i$.

`GEN FpX_invLaplace(GEN x, GEN p)` return $\sum_{i=0}^{n-1} x_i / i! X^i$.

7.3.7 `FpXQ, Fq`. Let `p` a `t_INT` and `T` an `FpX` for `p`, both to be given in the function arguments; an `FpXQ` object is an `FpX` whose degree is strictly less than the degree of `T`. An `Fq` is either an `FpXQ` or an `Fp`. Both represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, in which all operations below take place. In addition, `Fq` routines also allow `T` = `NULL`, in which case no reduction mod `T` is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for `gc_upto`). Besides `T` and `p`, arguments are either `FpXQ` or `Fq` depending on the function name. (All `Fq` routines accept `FpXQ`s by definition, not the other way round.)

7.3.7.1 Preconditioned reduction.

For faster reduction, the modulus T can be replaced by an extended modulus in all FpXQ- and Fq-classes functions, and in FpX_rem and FpX_divrem. An extended modulus(FpXT, which is a tree whose leaves are FpX)In current implementation, an extended modulus is either a plain modulus (an FpX) or a pair of polynomials, one being the plain modulus T and the other being FpX_invBarret(T, p).

GEN FpX_get_red(GEN T, GEN p) returns the extended modulus eT.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_FpX_mod(GEN eT) returns the underlying modulus T.

GEN get_FpX_var(GEN eT) returns the variable number varn(T).

GEN get_FpX_degree(GEN eT) returns the degree degpol(T).

7.3.7.2 Conversions.

int ff_parse_Tp(GEN Tp, GEN *T, GEN *p, long red) Tp is either a prime number p or a t_VEC with 2 entries T (an irreducible polynomial mod p) and p (a prime number). Sets *p and *T to the corresponding GENs (NULL if undefined). If red is nonzero, normalize *T as an FpX; on the other hand, to initialize a p-adic function, set red to 0 and *T is left as is and must be a ZX to start with. Return 1 on success, and 0 on failure. This helper routine is used by GP functions such as factormod where a single user argument defines a finite field. t_FFELT is not supported.

GEN Rg_is_FpXQ(GEN z, GEN *T, GEN *p), checks if z is a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything for which Rg_is_Fp return 1, a t_POL for which RgX_to_FpX return 1, a t_POLMOD whose modulus is equal to *T if *T is not NULL (once mapped to a FpX), or a t_FFELT z with the same definition field as *T if *T is not NULL and is a t_FFELT.

If an integer modulus is found it is put in *p, else *p is left unchanged. If a polynomial modulus is found it is put in *T, if a t_FFELT z is found, z is put in *T, else *T is left unchanged.

int RgX_is_FpXQX(GEN z, GEN *T, GEN *p), z a t_POL, checks if it can be mapped to a FpXQX, by checking Rg_is_FpXQ coefficientwise.

GEN Rg_to_FpXQ(GEN z, GEN T, GEN p), z a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything Rg_to_Fp can be applied to, a t_POL to which RgX_to_FpX can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a FpX), a suitable t_RFRAC. Returns z as an FpXQ, normalized.

GEN Rg_to_Fq(GEN z, GEN T, GEN p), applies Rg_to_Fp if T is NULL and Rg_to_FpXQ otherwise.

GEN RgX_to_FpXQX(GEN z, GEN T, GEN p), z a t_POL, returns the FpXQ obtained by applying Rg_to_FpXQ coefficientwise.

GEN RgX_to_FqX(GEN z, GEN T, GEN p): let z be a t_POL; returns the FqX obtained by applying Rg_to_Fq coefficientwise.

GEN Fq_to_FpXQ(GEN z, GEN T, GEN p /*unused*/) if z is a t_INT, convert it to a constant polynomial in the variable of T, otherwise return z (shallow function).

GEN Fq_red(GEN x, GEN T, GEN p), x a ZX or t_INT, reduce it to an Fq (T = NULL is allowed iff x is a t_INT).

GEN FqX_red(GEN x, GEN T, GEN p), x a t_POL whose coefficients are ZXs or t_INTs, reduce them to Fqs. (If T = NULL, as FpXX_red(x, p).)

GEN FqC_red(GEN x, GEN T, GEN p), x a vector of ZXs or t_INTs, reduce them to a t_COL of Fqs. (If T = NULL, only reduce components mod p to FpXs or Fps.)

GEN FqV_red(GEN x, GEN T, GEN p), x a vector of ZXs or t_INTs, reduce them to a t_VEC of Fqs. (If T = NULL, only reduce components mod p to FpXs or Fps.)

GEN FqM_red(GEN x, GEN T, GEN p), x a matrix of ZXs or t_INTs, reduce them to Fqs. (If T = NULL, only reduce components mod p to FpXs or Fps.)

GEN FpXQ_red(GEN x, GEN T, GEN p) x a t_POL whose coefficients are t_INTs, reduce them to FpXQs.

7.3.8 FpXQ.

GEN FpXQ_add(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sub(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sqr(GEN x, GEN T, GEN p)

GEN FpXQ_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_inv(GEN x, GEN T, GEN p) computes the inverse of x

GEN FpXQ_invsafe(GEN x, GEN T, GEN p), as FpXQ_inv, returning NULL if x is not invertible.

GEN FpXQ_pow(GEN x, GEN n, GEN T, GEN p) computes x^n.

GEN FpXQ_powu(GEN x, ulong n, GEN T, GEN p) computes x^n for small n.

In the following three functions the integer parameter ord can be given either as a positive t_INT N, or as its factorization matrix faN, or as a pair $[N, faN]$. The parameter may be omitted by setting it to NULL (the value is then $p^d - 1$, $d = \deg T$).

GEN FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let g be of order dividing ord in the finite field $\mathbf{F}_p[X]/(T)$, return e such that $a^e = g$. If e does not exists, the result is undefined. Assumes that T is irreducible mod p.

GEN Fp_FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ_log, a being a Fp.

GEN FpXQ_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ a. Assume that ord is a multiple of the order of a. Assume that T is irreducible mod p.

int FpXQ_issquare(GEN x, GEN T, GEN p) returns 1 if x is a square and 0 otherwise. Assumes that T is irreducible mod p.

GEN FpXQ_sqrt(GEN x, GEN T, GEN p) returns a square root of x. Return NULL if x is not a square.

GEN FpXQ_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn) Let Tbe irreducible mod p and $q = p^{\deg T}$; returns NULL if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if zn is not NULL set it to a primitive m-th root of 1 in $\mathbf{F}_q$, $m = \gcd(q-1, n)$ allowing to compute all m solutions in $\mathbf{F}_q$ of the equation $x^n = a$.

7.3.9 Fq.

`GEN random_Fq(GEN T, GEN p)` returns a random `Fq`

`GEN Fq_add(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_sub(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Fq_Fp_mul(GEN x, GEN y, GEN T, GEN p)` multiplies the `Fq` x by the `t_INT` y.

`GEN Fq_mulu(GEN x, ulong y, GEN T, GEN p)` multiplies the `Fq` x by the scalar y.

`GEN Fq_halve(GEN x, GEN T, GEN p)` returns z such that $2z = x$ assuming such z exists.

`GEN Fq_sqr(GEN x, GEN T, GEN p)`

`GEN Fq_neg(GEN x, GEN T, GEN p)`

`GEN Fq_neg_inv(GEN x, GEN T, GEN p)` computes $-\mathtt{x}^{-1}$

`GEN Fq_inv(GEN x, GEN pol, GEN p)` computes $\mathtt{x}^{-1}$, raising an error if `x` is not invertible.

`GEN Fq_invsafe(GEN x, GEN pol, GEN p)` as `Fq_inv`, but returns `NULL` if `x` is not invertible.

`GEN Fq_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FqV_inv(GEN x, GEN T, GEN p)` x being a vector of `Fq`s, return the vector of inverses of the $x[i]$. The routine uses Montgomery's trick, and involves a single inversion, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ `FpXQ` are left on stack, besides the N in the result.

`GEN FqV_factorback(GEN L, GEN e, GEN T, GEN p)` given an `FqV` L and a `ZV` or `zv` e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

`GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p)` returns $\mathtt{x}^{\mathtt{n}}$.

`GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p)` returns $\mathtt{x}^{\mathtt{n}}$ for small n.

`GEN Fq_log(GEN a, GEN g, GEN ord, GEN T, GEN p)` as `Fp_log` or `FpXQ_log`.

`int Fq_issquare(GEN x, GEN T, GEN p)` returns 1 if x is a square and 0 otherwise. Assumes that `T` is irreducible mod `p` and that p is prime; $T =$ `NULL` is forbidden unless x is an `Fp`.

`long Fq_ispower(GEN x, GEN n, GEN T, GEN p)` returns 1 if x is a n-th power and 0 otherwise. Assumes that `T` is irreducible mod `p` and that p is prime; $T =$ `NULL` is forbidden unless x is an `Fp`.

`GEN Fq_sqrt(GEN x, GEN T, GEN p)` returns a square root of `x`. Return `NULL` if `x` is not a square.

`GEN Fq_sqrtn(GEN a, GEN n, GEN T, GEN p, GEN *zn)` as `FpXQ_sqrtn`.

`GEN FpXQ_charpoly(GEN x, GEN T, GEN p)` returns the characteristic polynomial of `x`

`GEN FpXQ_minpoly(GEN x, GEN T, GEN p)` returns the minimal polynomial of `x`

`GEN FpXQ_norm(GEN x, GEN T, GEN p)` returns the norm of `x`

`GEN FpXQ_trace(GEN x, GEN T, GEN p)` returns the trace of `x`

`GEN FpXQ_conjvec(GEN x, GEN T, GEN p)` returns the vector of conjugates $[x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}]$ where n is the degree of T.

`GEN gener_FpXQ(GEN T, GEN p, GEN *po)` returns a primitive root modulo (T, p). T is an `FpX` assumed to be irreducible modulo the prime p. If `po` is not `NULL` it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

`GEN gener_FpXQ_local(GEN T, GEN p, GEN L)`, `L` being a vector of primes dividing $p^{\deg T} - 1$, returns an element of $G := \mathbf{F}_p[x]/(T)$ which is a generator of the ℓ-Sylow of G for every ℓ in `L`. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square if p is odd.

`GEN gener_Fq_local(GEN T, GEN p, GEN L)` as `pgener_Fp_local(p, L)` if T is `NULL`, or `gener_FpXQ_local` (otherwise).

`GEN FpXQ_powers(GEN x, long n, GEN T, GEN p)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ as a `t_VEC` of `FpXQ`s.

`GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p)`, as `FpXQ_powers`$(x, n-1, T, p)$, but returns the powers as a an $m \times n$ matrix. Usually, we have $m = n = \deg T$.

`GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p)` computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

`GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p)` a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[2]$.

`GEN FpXQ_auttrace(GEN a, ulong n, GEN T, GEN p)` a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[2]$.

`GEN FpXQ_autpowers(GEN S, long n, GEN T, GEN p)` returns $[x, S(x), S(S(x)), \ldots, S^{(n)}(x)]$ as a `t_VEC` of `FpXQ`s.

`GEN FpXQM_autsum(GEN a, long n, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[2]$ is a square matrix.

`GEN FpX_FpXQ_eval(GEN f, GEN x, GEN T, GEN p)` returns `f`(`x`).

`GEN FpX_FpXQV_eval(GEN f, GEN V, GEN T, GEN p)` returns `f`(`x`), assuming that `V` was computed by `FpXQ_powers`(`x`, n, `T`, `p`).

`GEN FpXC_FpXQ_eval(GEN C, GEN x, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the vector C and returns a `t_COL`.

`GEN FpXC_FpXQV_eval(GEN C, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the vector C and returns a `t_COL`.

`GEN FpXM_FpXQV_eval(GEN M, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the matrix M.

7.3.10 `FpXn`. Let `p` a `t_INT` and `T` an `FpX` for `p`, both to be given in the function arguments; an `FpXn` object is an `FpX` whose degree is strictly less than `n`. They represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, in which all operations below take place. They can be seen as truncated power series.

`GEN FpXn_mul(GEN x, GEN y, long n, GEN p)` return $xy \pmod{X^n}$.

`GEN FpXn_sqr(GEN x, long n, GEN p)` return $x^2 \pmod{X^n}$.

`GEN FpXn_div(GEN x, GEN y, long n, GEN p)` return $x/y \pmod{X^n}$.

`GEN FpXn_inv(GEN x, long n, GEN p)` return $1/x \pmod{X^n}$.

`GEN FpXn_exp(GEN f, long n, GEN p)` return $\exp(f)$ as a composition of formal power series. It is required that the valuation of f is positive and that $p > n$.

`GEN FpXn_expint(GEN f, long n, GEN p)` return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

7.3.11 `FpXC`, `FpXM`.

`GEN FpXC_center(GEN C, GEN p, GEN pov2)`

`GEN FpXM_center(GEN M, GEN p, GEN pov2)`

7.3.12 `FpXX`, `FpXY`. Contrary to what the name implies, an `FpXX` is a `t_POL` whose coefficients are either `t_INT`s or `FpX`s. This reduces memory overhead at the expense of consistency. The prefix `FpXY` is an alias for `FpXX` when variables matters.

`GEN FpXX_red(GEN z, GEN p)`, `z` a `t_POL` whose coefficients are either `ZX`s or `t_INT`s. Returns the `t_POL` equal to `z` with all components reduced modulo `p`.

`GEN FpXX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FpXX_add(GEN x, GEN y, GEN p)` adds `x` and `y`.

`GEN FpXX_sub(GEN x, GEN y, GEN p)` returns `x` − `y`.

`GEN FpXX_Fp_sub(GEN x, GEN y, GEN p)` substracts the `FpXX` `x` by the `Fp` `y`.

`GEN FpXX_neg(GEN x, GEN p)` returns −`x`.

`GEN FpXX_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the `Fp` `y`.

`GEN FpXX_FpX_mul(GEN x, GEN y, GEN p)` multiplies the coefficients of the `FpXX` `x` by the `FpX` `y`.

`GEN FpXX_mulu(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the scalar `y`.

`GEN FpXX_halve(GEN x, GEN p)` returns z such that $2z = x$ assuming such z exists.

`GEN FpXX_deriv(GEN P, GEN p)` differentiates `P` with respect to the main variable.

`GEN FpXX_integ(GEN P, GEN p)` returns the primitive of `P` with respect to the main variable whose constant term is 0.

`GEN FpXY_eval(GEN Q, GEN y, GEN x, GEN p)` Q being an `FpXY`, i.e. a `t_POL` with `Fp` or `FpX` coefficients representing an element of $\mathbf{F}_p[X][Y]$. Returns the `Fp` $Q(x, y)$.

`GEN FpXY_evalx(GEN Q, GEN x, GEN p)` Q being an `FpXY`, returns the `FpX` $Q(x, Y)$, where Y is the main variable of Q.

GEN FpXY_evaly(GEN Q, GEN y, GEN p, long vx) Q an FpXY, returns the FpX $Q(X, y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_FpXQ_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an FpXQ, returns the FpXQX $Q(X, y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_Fq_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an Fq, returns the FqX $Q(X, y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_FpXQ_evalx(GEN Q, GEN x, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x, Y)$, where Y is the first variable of Q.

GEN FpXY_FpXQV_evalx(GEN Q, GEN V, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x, Y)$, where Y is the first variable of Q, assuming that V was computed by FpXQ_powers(x, n, T, p).

GEN FpXYQQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x being a FpXY, T being a FpX and S being a FpY, return $x^n \pmod{S, T, p}$.

7.3.13 FpXk. An FpXk is a multivariate polynomial with t_INT coefficients, considered over $\mathbf{Z}/p\mathbf{Z}$.

GEN FpXk_red(GEN z, GEN p), z a ZXk, returns lift(z * Mod(1,p)), normalized.

GEN FpXk_gcd(GEN x, GEN y, GEN p) returns a gcd of the FpXk x and y.

int Rg_is_FpXk(GEN z, GEN *p) checks if z can be mapped to a FpXk by applying Rg_is_Fp recursively.

GEN Rg_to_FpXk(GEN z, GEN p) converts z to a FpXk by applying Rg_to_Fp recursively.

7.3.14 FpXQX, FqX. Contrary to what the name implies, an FpXQX is a t_POL whose coefficients are Fqs. So the only difference between FqX and FpXQX routines is that T = NULL is not allowed in the latter. (It was thought more useful to allow t_INT components than to enforce strict consistency, which would not imply any efficiency gain.)

7.3.14.1 Basic operations.

GEN FqX_add(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_add(GEN x, GEN y, GEN T, GEN p) adds the Fq y to the FqX x.

GEN FqX_Fq_sub(GEN x, GEN y, GEN T, GEN p) substracts the Fq y to the FqX x.

GEN FqX_neg(GEN x, GEN T, GEN p)

GEN FqX_sub(GEN x, GEN y, GEN T, GEN p)

GEN FqX_mul(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the Fq y.

GEN FqX_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the FqX x by the scalar y.

GEN FqX_halve(GEN x, GEN T, GEN p) returns z such that $2z = x$ assuming such z exists.

GEN FqX_Fp_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the t_INT y.

GEN FqX_Fq_mul_to_monic(GEN x, GEN y, GEN T, GEN p) returns xy assuming the result is monic of the same degree as x (in particular $y \neq 0$).

GEN FpXQX_normalize(GEN z, GEN T, GEN p)

GEN FqX_normalize(GEN z, GEN T, GEN p) divides the FqX z by its leading term. The leading coefficient becomes 1 as a t_INT.

GEN FqX_sqr(GEN x, GEN T, GEN p)

GEN FqX_powu(GEN x, ulong n, GEN T, GEN p)

GEN FqX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *z)

GEN FqX_div(GEN x, GEN y, GEN T, GEN p)

GEN FqX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)

GEN FqX_rem(GEN x, GEN y, GEN T, GEN p)

GEN FqX_deriv(GEN x, GEN T, GEN p) returns the derivative of x. Not stack-clean but suitable for gc_upto.

GEN FqX_integ(GEN x, GEN T, GEN p) returns the primitive of x. whose constant term is 0.

GEN FqX_Fq_translate(GEN P, GEN c, GEN T, GEN p) let c be an Fq defined modulo (p, T), and let P be an FqX; returns the translated FqX of $P(X + c)$.

GEN FqX_gcd(GEN P, GEN Q, GEN T, GEN p) returns a (not necessarily monic) greatest common divisor of x and y.

GEN FqX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv) returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets *u, *v to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$.

GEN FqX_halfgcd(GEN x, GEN y, GEN T, GEN p) returns a two-by-two FqXM M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FqX_eval(GEN x, GEN y, GEN T, GEN p) evaluates the FqX x at the Fq y. The result is an Fq.

GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p) Q an FqXY, i.e. a t_POL with Fq or FqX coefficients representing an element of $\mathbf{F}_q[X][Y]$. Returns the Fq $Q(x, y)$.

GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p) Q being an FqXY, returns the FqX $Q(x, Y)$, where Y is the main variable of Q.

GEN random_FpXQX(long d, long v, GEN T, GEN p) returns a random FpXQX in variable v, of degree less than d.

GEN FpXQX_renormalize(GEN x, long lx)

GEN FpXQX_red(GEN z, GEN T, GEN p) z a t_POL whose coefficients are ZXs or t_INTs, reduce them to FpXQs.

GEN FpXQXV_red(GEN z, GEN T, GEN p), z a t_VEC of ZXX. Applies FpX_red componentwise and returns the result (and we obtain a vector of FpXQXs).

GEN FpXQXT_red(GEN z, GEN T, GEN p), z a tree of ZXX. Applies FpX_red to each leaf and returns the result (and we obtain a tree of FpXQXs).

GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)

GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p). Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from

multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P (see `RgXX_to_Kronecker`), this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t)$ mod $(p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_sqr(GEN x, GEN T, GEN p)`

`GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)`

`GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)`

`GEN FpXQX_rem(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_powu(GEN x, ulong n, GEN T, GEN p)` returns x^n.

`GEN FpXQX_digits(GEN x, GEN B, GEN T, GEN p)`

`GEN FpXQX_dotproduct(GEN x, GEN y, GEN T, GEN p)` returns the scalar product of the coefficients of x and y.

`GEN FpXQXV_FpXQX_fromdigits(GEN v, GEN B, GEN T, GEN p)`

`GEN FpXQX_invBarrett(GEN y, GEN T, GEN p)` returns the Barrett inverse of the `FpXQX` y, namely a lift of $1/$`polrecip`$(y) + O(x^{\deg(y)-1})$.

`GEN FpXQXV_prod(GEN V, GEN T, GEN p)`, `V` being a vector of `FpXQX`, returns their product.

`GEN FpXQX_gcd(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)`

`GEN FpXQX_halfgcd(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_halfgcd_all(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)`

`GEN FpXQX_resultant(GEN x, GEN y, GEN T, GEN p)` returns the resultant of `x` and `y`.

`GEN FpXQX_disc(GEN x, GEN T, GEN p)` returns the discriminant of `x`.

`GEN FpXQX_FpXQXQ_eval(GEN f, GEN x, GEN S, GEN T, GEN p)` returns `f(x)`.

`GEN FpXQX_FpXQ_translate(GEN P, GEN c, GEN T, GEN p)`

7.3.15 `FpXQXn`, `FqXn`.

A `FpXQXn` is a `t_FpXQX` which represents an element of the ring $(Fp[X]/T(X))[Y]/(Y^n)$, where T is a `FpX`.

`GEN FpXQXn_sqr(GEN x, long n, GEN T, GEN p)`

`GEN FqXn_sqr(GEN x, long n, GEN T, GEN p)`

`GEN FpXQXn_mul(GEN x, GEN y, long n, GEN T, GEN p)`

`GEN FqXn_mul(GEN x, GEN y, long n, GEN T, GEN p)`

`GEN FpXQXn_div(GEN x, GEN y, long n, GEN T, GEN p)`

`GEN FpXQXn_inv(GEN x, long n, GEN T, GEN p)`

GEN FqXn_inv(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_exp(GEN x, long n, GEN T, GEN p) return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

GEN FqXn_exp(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_expint(GEN f, long n, GEN p) return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

GEN FqXn_expint(GEN x, long n, GEN T, GEN p)

7.3.16 FpXQXQ, FqXQ.

A FpXQXQ is a t_FpXQX which represents an element of the ring $(Fp[X]/T(X))[Y]/S(X,Y)$, where T is a FpX and S a FpXQX modulo T. A FqXQ is identical except that T is allowed to be NULL in which case S must be a FpX.

7.3.16.1 Preconditioned reduction.

For faster reduction, the modulus S can be replaced by an extended modulus, which is an FpXQXT, in all FpXQXQ- and FqXQ-classes functions, and in FpXQX_rem and FpXQX_divrem.

GEN FpXQX_get_red(GEN S, GEN T, GEN p) returns the extended modulus eS.

GEN FqX_get_red(GEN S, GEN T, GEN p) identical, but allow T to be NULL, in which case it returns FpX_get_red(S,p).

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_FpXQX_mod(GEN eS) returns the underlying modulus S.

GEN get_FpXQX_var(GEN eS) returns the variable number of the modulus.

GEN get_FpXQX_degree(GEN eS) returns the degree of the modulus.

Furthermore, ZXXT_to_FlxXT allows to convert an extended modulus for a FpXQX to an extended modulus for the corresponding FlxqX.

7.3.16.2 basic operations.

GEN FpXQX_FpXQXQV_eval(GEN f, GEN V, GEN S, GEN T, GEN p) returns f(x), assuming that V was computed by FpXQXQ_powers(x, n, S, T, p).

GEN FpXQXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns $x * y^{-1}$ modulo S.

GEN FpXQXQ_inv(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^{-1} modulo S.

GEN FpXQXQ_invsafe(GEN x, GEN S, GEN T, GEN p), as FpXQXQ_inv, returning NULL if x is not invertible.

GEN FpXQXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns xy modulo S.

GEN FpXQXQ_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^2 modulo S.

GEN FpXQXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^n modulo S.

`GEN FpXQXQ_powers(GEN x, long n, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $[\mathtt{x}^0, \ldots, \mathtt{x}^n]$ as a `t_VEC` of `FpXQXQ`s.

`GEN FpXQXQ_halfFrobenius(GEN A, GEN S, GEN T, GEN p)` returns $A(X)^{(q-1)/2} \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

`GEN FpXQXQ_minpoly(GEN x, GEN S, GEN T, GEN p)`, as `FpXQ_minpoly`

`GEN FpXQXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p)` returns the same powers of `x` as `FpXQXQ_powers`$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

`GEN FpXQXQ_autpow(GEN a, long n, GEN S, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns $[\sigma^n(X), \sigma^n(Y)]$.

`GEN FpXQXQ_autsum(GEN a, long n, GEN S, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[3]$.

`GEN FpXQXQ_auttrace(GEN a, long n, GEN S, GEN T, GEN p)` σ being the automorphism defined by $\sigma(X) = X \pmod{T(X)}$, $\sigma(Y) = a[1] \pmod{S(X,Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[2]$.

`GEN FqXQ_add(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `x + y` modulo `S`.

`GEN FqXQ_sub(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `x - y` modulo `S`.

`GEN FqXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FqX`s, returns `xy` modulo `S`.

`GEN FqXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns `x/y` modulo `S`.

`GEN FqXQ_inv(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns $\mathtt{x}^{-1}$ modulo `S`.

`GEN FqXQ_invsafe(GEN x, GEN S, GEN T, GEN p)` , as `FqXQ_inv`, returning `NULL` if `x` is not invertible.

`GEN FqXQ_sqr(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns $\mathtt{x}^2$ modulo `S`.

`GEN FqXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns $\mathtt{x}^n$ modulo `S`.

`GEN FqXQ_powers(GEN x, long n, GEN S, GEN T, GEN p)`, `x` and `S` being `FqX`s, returns $[\mathtt{x}^0, \ldots, \mathtt{x}^n]$ as a `t_VEC` of `FqXQ`s.

`GEN FqXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p)` returns the same powers of `x` as `FqXQ_powers`$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

`GEN FqV_roots_to_pol(GEN V, GEN T, GEN p, long v)`, `V` being a vector of `Fq`s, returns the monic `FqX` $\prod_i (\mathtt{pol\_x}[\mathtt{v}] - \mathtt{V}[\mathtt{i}])$.

7.3.16.3 Miscellaneous operations.

`GEN init_Fq(GEN p, long n, long v)` returns an irreducible polynomial of degree `n` > 0 over $\mathbf{F}_p$, in variable `v`.

`int FpXQX_is_squarefree(GEN P, GEN T, GEN p)` returns 1 if P is squarefree, 0 otherwise.

`int FqX_is_squarefree(GEN P, GEN T, GEN p)`

`GEN FpXQX_roots(GEN f, GEN T, GEN p)` return the roots of f in $\mathbf{F}_p[X]/(T)$. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FpXQX_roots_mult(GEN f, long n, GEN T, GEN p)` returns the roots in **Z/pZ** with multiplicity at least n of the `FpXQX f` (without multiplicity, as a vector of `FpXQ`s). Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FqX_roots(GEN f, GEN T, GEN p)` same but allow `T` $=$ `NULL`.

`GEN FpXQX_factor(GEN f, GEN T, GEN p)` same output convention as `FpX_factor`. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FqX_factor(GEN f, GEN T, GEN p)` same but allow `T` $=$ `NULL`.

`GEN FpXQX_factor_squarefree(GEN f, GEN T, GEN p)` squarefree factorization of f modulo (T, p); same output convention as `FpX_factor_squarefree`. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FqX_factor_squarefree(GEN f, GEN T, GEN p)` same but allow `T` $=$ `NULL`.

`GEN FpXQX_ddf(GEN f, GEN T, GEN p)` as `FpX_ddf`.

`GEN FqX_ddf(GEN f, GEN T, GEN p)` same but allow `T` $=$ `NULL`.

`long FpXQX_ddf_degree(GEN f, GEN XP, GEN T, GEN p)`, as `FpX_ddf_degree`.

`GEN FpXQX_degfact(GEN f, GEN T, GEN p)`, as `FpX_degfact`.

`GEN FqX_degfact(GEN f, GEN T, GEN p)` same but allow `T` $=$ `NULL`.

`GEN FpXQX_split_part(GEN f, GEN T, GEN p)` returns the largest totally split squarefree factor of f.

`long FpXQX_ispower(GEN f, ulong k, GEN T, GEN p, GEN *pt)` return 1 if the `FpXQX` f is a k-th power, 0 otherwise. If `pt` is not `NULL`, set it to g such that $g^k = f$.

`long FqX_ispower(GEN f, ulong k, GEN T, GEN p, GEN *pt)` same but allow `T` $=$ `NULL`.

`GEN FpX_factorff(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Factor the `FpX P` over the finite field $\mathbf{F}_p[Y]/(T(Y))$. See `FpX_factorff_irred` if `P` is known to be irreducible of $\mathbf{F}_p$.

`GEN FpX_rootsff(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Returns the roots of the `FpX P` belonging to the finite field $\mathbf{F}_p[Y]/(T(Y))$.

`GEN FpX_factorff_irred(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Factors the *irreducible* `FpX P` over the finite field $\mathbf{F}_p[Y]/(T(Y))$ and returns the vector of irreducible `FqX`s factors (the exponents, being all equal to 1, are not included).

`GEN FpX_ffisom(GEN P, GEN Q, GEN p)`. Assumes `p` prime, `P`, `Q` are `ZX`s, both irreducible mod `p`, and $\deg(P) \mid \deg Q$. Outputs a monomorphism between $\mathbf{F}_p[X]/(P)$ and $\mathbf{F}_p[X]/(Q)$, as a polynomial R such that `Q` $\mid$ `P`(R) in $\mathbf{F}_p[X]$. If `P` and `Q` have the same degree, it is of course an isomorphism.

`void FpX_ffintersect(GEN P, GEN Q, long n, GEN p, GEN *SP, GEN *SQ, GEN MA, GEN MB)` Assumes `p` is prime, `P`, `Q` are `ZX`s, both irreducible mod `p`, and `n` divides both the degree of `P` and `Q`. Compute `SP` and `SQ` such that the subfield of $\mathbf{F}_p[X]/(P)$ generated by `SP` and the subfield of $\mathbf{F}_p[X]/(Q)$ generated by `SQ` are isomorphic of degree `n`. The polynomials `P` and `Q` do not need to be of the same variable. If `MA` (resp. `MB`) is not `NULL`, it must be the matrix of the Frobenius map in $\mathbf{F}_p[X]/(P)$ (resp. $\mathbf{F}_p[X]/(Q)$).

`GEN FpXQ_ffisom_inv(GEN S, GEN T, GEN p)`. Assumes `p` is prime, `T` a `ZX`, which is irreducible modulo `p`, `S` a `ZX` representing an automorphism of $\mathbf{F}_q := \mathbf{F}_p[X]/(\mathtt{T})$. ($\mathtt{S}(X)$ is the image of X by the automorphism.) Returns the inverse automorphism of `S`, in the same format, i.e. an `FpX` H such that $H(\mathtt{S}) \equiv X$ modulo $(\mathtt{T}, \mathtt{p})$.

`long FpXQX_nbfact(GEN S, GEN T, GEN p)` returns the number of irreducible factors of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FpXQX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, GEN p)` as `FpXQX_nbfact` where `Xq` is `FpXQX_Frobenius(S, T, p)`.

`long FqX_nbfact(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`long FpXQX_nbroots(GEN S, GEN T, GEN p)` returns the number of roots of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FqX_nbroots(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`GEN FpXQX_Frobenius(GEN S, GEN T, GEN p)` returns $X^q \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

7.3.17 `Flx`. Let `p` be an `ulong`, not assumed to be prime unless mentioned otherwise (e.g., all functions involving Euclidean divisions and factorizations), to be given the function arguments; an `Fl` is an `ulong` belonging to $[0, \mathtt{p} - 1]$, an `Flx z` is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn(`v`)` for some variable v, then the coefficients by increasing degree. An `FlxX` is a `t_POL` whose coefficients are `Flx`s.

In the following, an argument called `sv` is of the form `evalvarn`(v) for some variable number v.

7.3.17.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus (`FlxT`) in all `Flxq`-classes functions, and in `Flx_divrem`.

`GEN Flx_get_red(GEN T, ulong p)` returns the extended modulus `eT`.

`GEN Flx_get_red_pre(GEN T, ulong p, ulong pi)` as `Flx_get_red`. We assume pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_Flx_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_Flx_var(GEN eT)` returns the variable number of the modulus.

`GEN get_Flx_degree(GEN eT)` returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

7.3.17.2 Basic operations.

In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`ulong Flx_lead(GEN x)` returns the leading coefficient of x as a `ulong` (return 0 for the zero polynomial).

`ulong Flx_constant(GEN x)` returns the constant coefficient of x as a `ulong` (return 0 for the zero polynomial).

`GEN Flx_red(GEN z, ulong p)` converts from `zx` with nonnegative coefficients to `Flx` (by reducing them mod `p`).

`int Flx_equal1(GEN x)` returns 1 (true) if the `Flx` x is equal to 1, 0 (false) otherwise.

`int Flx_equal(GEN x, GEN y)` returns 1 (true) if the `Flx` x and y are equal, and 0 (false) otherwise.

`GEN Flx_copy(GEN x)` returns a copy of `x`.

`GEN Flx_add(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_add(GEN y, ulong x, ulong p)`

`GEN Flx_neg(GEN x, ulong p)`

`GEN Flx_neg_inplace(GEN x, ulong p)`, same as `Flx_neg`, in place (`x` is destroyed).

`GEN Flx_sub(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_sub(GEN y, ulong x, ulong p)`

`GEN Flx_halve(GEN x, ulong p)` returns z such that $2z = x$ modulo p assuming such z exists.

`GEN Flx_mul(GEN x, GEN y, ulong p)`

`GEN Flx_mul_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_Fl_mul(GEN y, ulong x, ulong p)`

`GEN Flx_Fl_mul_pre(GEN y, ulong x, ulong p, ulong pi)`

`GEN Flx_double(GEN y, ulong p)` returns $2y$.

`GEN Flx_triple(GEN y, ulong p)` returns $3y$.

`GEN Flx_mulu(GEN y, ulong x, ulong p)` as `Flx_Fl_mul` but do not assume that $x < p$.

`GEN Flx_Fl_mul_to_monic(GEN y, ulong x, ulong p)` returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

`GEN Flx_sqr(GEN x, ulong p)`

`GEN Flx_sqr_pre(GEN x, ulong p, ulong pi)`

`GEN Flx_powu(GEN x, ulong n, ulong p)` return x^n.

`GEN Flx_powu_pre(GEN x, ulong n, ulong p, ulong pi)`

`GEN Flx_convol(GEN x, GEN y, ulong p)` return the-term by-term product of x and y.

`GEN Flx_divrem(GEN x, GEN y, ulong p, GEN *pr)`, here p must be prime.

`GEN Flx_divrem_pre(GEN x, GEN y, ulong p, ulong pi, GEN *pr)`

`GEN Flx_div(GEN x, GEN y, ulong p)`, here p must be prime.

`GEN Flx_div_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_rem(GEN x, GEN y, ulong p)`, here p must be prime.

`GEN Flx_rem_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_deriv(GEN z, ulong p)`

`GEN Flx_integ(GEN z, ulong p)`, here p must be prime.

`GEN Flx_Fl_translate(GEN P, ulong c, ulong p)` return $P(x+c)$, p must be prime. Asymptotically fast (quasi-linear in the degree of P).

`GEN Flx_translate1(GEN P, ulong p)` return $P(x+1)$, p must be prime. Asymptotically fast (quasi-linear in the degree of P).

`GEN Flx_translate1_basecase(GEN P, ulong p)` return $P(x+1)$, p need not be prime. Not asymptotically fast (quadratic in the degree of P).

`GEN zlx_translate1(GEN P, ulong p, long e)` return $P(x+1)$ modulo p^e for prime p. Asymptotically fast (quasi-linear in the degree of P).

`GEN Flx_diff1(GEN P, ulong p)` return $P(x+1) - P(x)$; p must be prime.

`GEN Flx_digits(GEN x, GEN B, ulong p)` returns a vector of `Flx` $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

`GEN FlxV_Flx_fromdigits(GEN v, GEN B, ulong p)` where $v = [c_0, \dots, c_n]$ is a vector of `Flx`, returns $\sum_{i=0}^{n} c_i B^i$.

`GEN Flx_Frobenius(GEN T, ulong p)` here p must be prime.

`GEN Flx_Frobenius_pre(GEN T, ulong p, ulong pi)`

`GEN Flx_matFrobenius(GEN T, ulong p)` here p must be prime.

`GEN Flx_matFrobenius_pre(GEN T, ulong p, ulong pi)`

`GEN Flx_gcd(GEN a, GEN b, ulong p)` returns a (not necessarily monic) greatest common divisor of x and y. Here p must be prime.

`GEN Flx_gcd_pre(GEN a, GEN b, ulong p)`

`GEN Flx_halfgcd(GEN x, GEN y, ulong p)` returns a two-by-two `FlxM` M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$. Assumes that p is prime.

`GEN Flx_halfgcd_pre(GEN a, GEN b, ulong p)`

`GEN Flx_halfgcd_all(GEN x, GEN y, ulong p, GEN *pt_a, GEN *pt_b)`

`GEN Flx_halfgcd_all_pre(GEN x, GEN y, ulong p, GEN *pt_a, GEN *pt_b)`

`GEN Flx_extgcd(GEN a, GEN b, ulong p, GEN *ptu, GEN *ptv)`, here p must be prime.

`GEN Flx_extgcd_pre(GEN a, GEN b, ulong p, ulong pi, GEN *ptu, GEN *ptv)`

`GEN Flx_roots(GEN f, ulong p)` returns the vector of roots of f (without multiplicity, as a `t_VECSMALL`). Assumes that p is prime.

`GEN Flx_roots_pre(GEN f, ulong p, ulong pi)`

`ulong Flx_oneroot(GEN f, ulong p)` returns one root $0 \leq r < p$ of the `Flx f` in $\mathbf{Z}/p\mathbf{Z}$. Return p if no root exists. Assumes that p is prime.

`GEN Flx_oneroot_pre(GEN f, ulong p)`, as `Flx_oneroot`

`ulong Flx_oneroot_split(GEN f, ulong p)` as `Flx_oneroot` but assume f is totally split. Assumes that p is prime.

`ulong Flx_oneroot_split_pre(GEN f, ulong p, ulong pi)`

`long Flx_ispower(GEN f, ulong k, ulong p, GEN *pt)` return 1 if the `Flx` f is a k-th power, 0 otherwise. If `pt` is not `NULL`, set it to g such that $g^k = f$.

`GEN Flx_factor(GEN f, ulong p)` Assumes that p is prime.

`GEN Flx_ddf(GEN f, ulong p)` Assumes that p is prime.

`GEN Flx_ddf_pre(GEN f, ulong p, ulong pi)`

`GEN Flx_factor_squarefree(GEN f, ulong p)` returns the squarefree factorization of f modulo p. This is a vector $[u_1, \ldots, u_k]$ of pairwise coprime `Flx` such that $u_k \neq 1$ and $f = \prod u_i^i$. Shallow function. Assumes that p is prime.

`GEN Flx_factor_squarefree_pre(GEN f, ulong p, ulong pi)`

`GEN Flx_mod_Xn1(GEN T, ulong n, ulong p)` return T modulo $(X^n + 1, p)$. Shallow function.

`GEN Flx_mod_Xnm1(GEN T, ulong n, ulong p)` return T modulo $(X^n - 1, p)$. Shallow function.

`GEN Flx_degfact(GEN f, ulong p)` as `FpX_degfact`. Assumes that p is prime.

`GEN Flx_factorff_irred(GEN P, GEN Q, ulong p)` as `FpX_factorff_irred`. Assumes that p is prime.

`GEN Flx_rootsff(GEN P, GEN T, ulong p)` as `FpX_rootsff`. Assumes that p is prime.

`GEN Flx_factcyclo(ulong n, ulong p, ulong m)` returns the factors of the n-th cyclotomic polynomial over `Fp`. if $m = 1$ returns a single factor.

`GEN Flx_ffisom(GEN P, GEN Q, ulong l)` as `FpX_ffisom`. Assumes that p is prime.

7.3.17.3 Miscellaneous operations.

`GEN pol0_Flx(long sv)` returns a zero `Flx` in variable v.

`GEN zero_Flx(long sv)` alias for `pol0_Flx`

`GEN pol1_Flx(long sv)` returns the unit `Flx` in variable v.

`GEN polx_Flx(long sv)` returns the variable v as degree 1 `Flx`.

`GEN polxn_Flx(long n, long sv)` Returns the monomial of degree n as a `Flx` in variable v; assume that $n \geq 0$.

`GEN monomial_Flx(ulong a, long d, long sv)` returns the `Flx` aX^d in variable v.

`GEN init_Flxq(ulong p, long n, long sv)` returns an irreducible polynomial of degree `n` > 0 over $\mathbf{F}_p$, in variable `v`.

`GEN Flx_normalize(GEN z, ulong p)`, as `FpX_normalize`.

`GEN Flx_rescale(GEN P, ulong h, ulong p)` returns $h^{\deg(P)}P(x/h)$, `P` is a `Flx` and `h` is a nonzero integer.

`GEN Flx_unscale(GEN P, ulong h, ulong p)` returns $P(hx)$, `P` is a `Flx` and `h` is a nonzero integer.

`GEN random_Flx(long d, long sv, ulong p)` returns a random `Flx` in variable `v`, of degree less than `d`.

`GEN Flx_recip(GEN x)`, returns the reciprocal polynomial

`ulong Flx_resultant(GEN a, GEN b, ulong p)`, returns the resultant of `a` and `b`. Assumes that p is prime.

`ulong Flx_resultant_pre(GEN a, GEN b, ulong p, ulong pi)`

`ulong Flx_extresultant(GEN a, GEN b, ulong p, GEN *ptU, GEN *ptV)` given two `Flx` `a` and `b`, returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set). Assumes that p is prime.

`ulong Flx_extresultant_pre(GEN a, GEN b, ulong p, ulong pi, GEN *ptU, GEN *ptV)`

`GEN Flx_composedprod(GEN P, GEN Q, ulong p)` if $P = a\prod_{i=1}^{m}(x-p_i)$ and $Q = b\prod_{j=1}^{n}(x-q_j)$ in some suitable algebraic extension, return $a^n b^m \prod_{i,j}(x-p_i q_j)$.

`GEN Flx_composedsum(GEN P, GEN Q, ulong p)` if $P = a\prod_{i=1}^{m}(x-p_i)$ and $Q = b\prod_{j=1}^{n}(x-q_j)$ in some suitable algebraic extension, return $a^n b^m \prod_{i,j}(x-(p_i+q_j))$.

`GEN Flx_invBarrett(GEN T, ulong p)`, returns the Barrett inverse M of T defined by $M(x)\times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T. Assumes that p is prime.

`GEN Flx_renormalize(GEN x, long l)`, as `FpX_renormalize`, where `l` = `lg(x)`, in place.

`GEN Flx_shift(GEN T, long n)` returns `T` $* x^n$ if $n \geq 0$, and `T`$\backslash x^{-n}$ otherwise.

`long Flx_val(GEN x)` returns the valuation of `x`, i.e. the multiplicity of the 0 root.

`long Flx_valrem(GEN x, GEN *Z)` as `RgX_valrem`, returns the valuation of `x`. In particular, if the valuation is 0, set `*Z` to x, not a copy.

`GEN Flx_div_by_X_x(GEN A, ulong a, ulong p, ulong *rem)`, returns the Euclidean quotient of the `Flx` `A` by $X -$ `a`, and sets `rem` to the remainder `A`(`a`).

`ulong Flx_eval(GEN x, ulong y, ulong p)`, as `FpX_eval`.

`ulong Flx_eval_pre(GEN x, ulong y, ulong p, ulong pi)`

`ulong Flx_eval_powers_pre(GEN P, GEN y, ulong p, ulong pi)`. Let y be the `t_VECSMALL` $(1, a, \ldots, a^n)$, where n is the degree of the `Flx` P, return $P(a)$.

`GEN Flx_Flv_multieval(GEN P, GEN v, ulong p)` returns the vector $[P(v[1]), \ldots, P(v[n])]$ as a `Flv`.

`ulong Flx_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of the coefficients of x and y.

`ulong Flx_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)`.

`GEN Flx_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q.

`GEN Flx_inflate(GEN P, long d)` returns $P(X^d)$.

`GEN Flx_splitting(GEN P, long k)`, as `RgX_splitting`.

`GEN Flx_blocks(GEN P, long n, long m)`, as `RgX_blocks`.

`int Flx_is_squarefree(GEN z, ulong p)`. Assumes that p is prime.

`int Flx_is_irred(GEN f, ulong p)`, as `FpX_is_irred`. Assumes that p is prime.

`int Flx_is_totally_split(GEN f, ulong p)` returns 1 if the `Flx f` splits into a product of distinct linear factors, 0 otherwise. Assumes that `p` is prime.

`int Flx_is_smooth(GEN f, long r, ulong p)` return 1 if all irreducible factors of f are of degree at most r, 0 otherwise. Assumes that p is prime.

`int Flx_is_smooth_pre(GEN f, long r, ulong p, ulong pi)`

`long Flx_nbroots(GEN f, ulong p)`, as `FpX_nbroots`. Assumes that p is prime.

`long Flx_nbfact(GEN z, ulong p)`, as `FpX_nbfact`. Assumes that p is prime.

`long Flx_nbfact_pre(GEN z, ulong p, ulong pi)`

`long Flx_nbfact_Frobenius(GEN f, GEN XP, ulong p)`, as `FpX_nbfact_Frobenius`. Assumes that p is prime.

`long Flx_nbfact_Frobenius_pre(GEN f, GEN XP, ulong p, ulong pi)`

`GEN Flx_degfact(GEN f, ulong p)`, as `FpX_degfact`. Assumes that p is prime.

`GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p)` Assume that the `Flx` z is squarefree mod the prime p. Returns a `t_VECSMALL` D with $\deg z$ entries, such that $D[i]$ is the number of irreducible factors of degree i. Set `nb` to the total number of irreducible factors (the sum of the $D[i]$). Assumes that p is prime.

`void Flx_ffintersect(GEN P, GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA, GEN MB)`
,
as `FpX_ffintersect`. Assumes that p is prime.

`GEN Flx_Laplace(GEN x, ulong p)`

`GEN Flx_invLaplace(GEN x, ulong p)`

`GEN Flx_Newton(GEN x, long n, ulong p)`

`GEN Flx_fromNewton(GEN x, ulong p)`

`GEN Flx_Teichmuller(GEN P, ulong p, long n)` Return a `ZX` Q such that $P \equiv Q \pmod{p}$ and $Q(X^p) = 0 \pmod{Q, p^n}$. Assumes that p is prime.

`GEN Flv_polint(GEN x, GEN y, ulong p, long sv)` as `FpV_polint`, returning an `Flx` in variable v. Assumes that p is prime.

`GEN Flv_Flm_polint(GEN x, GEN V, ulong p, long sv)` equivalent (but faster) to applying `Flv_polint(x,...)` to all the elements of the vector V (thus, returns a `FlxV`). Assumes that p is prime.

`GEN Flv_invVandermonde(GEN L, ulong d, ulong p)` L being a `Flv` of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, multiplied by `d`. If A is a `Flv` and $B = MA$, then the polynomial $P = \sum_{i=1}^{n} B[i]X^{i-1}$ verifies $P(L[i]) = dA[i]$ for $1 \le i \le n$. Assumes that p is prime.

`GEN Flv_roots_to_pol(GEN a, ulong p, long sv)` as `FpV_roots_to_pol` returning an `Flx` in variable v.

7.3.18 `FlxV`. See `FpXV` operations.

`GEN FlxV_Flc_mul(GEN V, GEN W, ulong p)`, as `FpXV_FpC_mul`.

`GEN FlxV_red(GEN V, ulong p)` reduces each components with `Flx_red`.

`GEN FlxV_prod(GEN V, ulong p)`, `V` being a vector of `Flx`, returns their product.

`GEN FlxV_factorback(GEN L, GEN e, ulong p, long v)` returns $\prod_i L_i^{e_i}$ where L is a vector of `Flx`s in the shifted variable `v` and e a vector of non-negative `t_INT`s or a `t_VECSMALL`.

`GEN FlxV_composedsum(GEN V, ulong p)`, `V` being a vector of `Flx`, returns their composed sum, see `Flx_composedsum`.

`ulong FlxC_eval_powers_pre(GEN x, GEN y, ulong p, ulong pi)` apply `Flx_eval_powers_pre` to all elements of `x`.

`GEN FlxV_Flv_multieval(GEN F, GEN v, ulong p)` assuming F is a vector of `Flx` with m entries and v is a `Flv` with m entries, returns the n-components vector (`FlvV`) whose j-th entry is $[F_j(v[1]), \dots, F_j(v[n])]$, with $F_j = F[j]$.

`GEN FlxC_neg(GEN x, ulong p)`

`GEN FlxC_sub(GEN x, GEN y, ulong p)`

`GEN FlxC_Fl_translate(GEN x, ulong c, ulong p)`

`GEN zero_FlxC(long n, long sv)`

7.3.19 `FlxM`. See `FpXM` operations.

`ulong FlxM_eval_powers_pre(GEN M, GEN y, ulong p, ulong pi)` this function applies `FlxC_eval_powers_pre` to all entries of `M`.

`GEN FlxM_neg(GEN x, ulong p)`

`GEN FlxM_sub(GEN x, GEN y, ulong p)`

`GEN FlxM_mul(GEN x, GEN y, ulong p)`

`GEN FlxM_Fl_translate(GEN x, ulong c, ulong p)`

`GEN zero_FlxM(long r, long c, long sv)`

`GEN zero_FlxM_copy(long r, long c, long sv)`

7.3.20 `FlxT`. See `FpXT` operations.

`GEN FlxT_red(GEN V, ulong p)` reduces each leaf with `Flx_red`.

7.3.21 `Flxn`. See `FpXn` operations. In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flxn_mul(GEN a, GEN b, long n, ulong p)` returns ab modulo X^n.

`GEN Flxn_mul_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_sqr(GEN a, long n, ulong p)` returns a^2 modulo X^n.

`GEN Flxn_sqr_pre(GEN a, long n, ulong p, ulong pi)`

`GEN Flxn_inv(GEN a, long n, ulong p)` returns $1/a$ modulo X^n.

`GEN Flxn_div(GEN a, GEN b, long n, ulong p)` returns a/b modulo X^n.

`GEN Flxn_div_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_red(GEN a, long n)` returns a modulo X^n.

`GEN Flxn_exp(GEN x, long n, ulong p)` return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

`GEN Flxn_expint(GEN f, long n, ulong p)` return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

7.3.22 `Flxq`. See `FpXQ` operations. In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flxq_add(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sub(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_sqr(GEN y, GEN T, ulong p)`

`GEN Flxq_sqr_pre(GEN y, GEN T, ulong p)`

`GEN Flxq_inv(GEN x, GEN T, ulong p)`

`GEN Flxq_inv_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_invsafe(GEN x, GEN T, ulong p)`

`GEN Flxq_invsafe_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_div(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_div_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_pow(GEN x, GEN n, GEN T, ulong p)`

`GEN Flxq_pow_pre(GEN x, GEN n, GEN T, ulong p, ulong pi)`

`GEN Flxq_powu(GEN x, ulong n, GEN T, ulong p)`

`GEN Flxq_powu_pre(GEN x, ulong n, GEN T, ulong p)`

`GEN FlxqV_factorback(GEN L, GEN e, GEN Tp, ulong p)`

`GEN Flxq_pow_init(GEN x, GEN n, long k, GEN T, ulong p)`

GEN Flxq_pow_init_pre(GEN x, GEN n, long k, GEN T, ulong p, ulong pi)

GEN Flxq_pow_table(GEN R, GEN n, GEN T, ulong p)

GEN Flxq_pow_table_pre(GEN R, GEN n, GEN T, ulong p, ulong pi)

GEN Flxq_powers(GEN x, long n, GEN T, ulong p)

GEN Flxq_powers_pre(GEN x, long n, GEN T, ulong p, ulong pi)

GEN Flxq_matrix_pow(GEN x, long m, long n, GEN T, ulong p), see FpXQ_matrix_pow.

GEN Flxq_matrix_pow_pre(GEN x, long m, long n, GEN T, ulong p, ulong pi)

GEN Flxq_autpow(GEN a, long n, GEN T, ulong p) see FpXQ_autpow.

GEN Flxq_autpow_pre(GEN a, long n, GEN T, ulong p, ulong pi)

GEN Flxq_autpowers(GEN a, long n, GEN T, ulong p) return $[X, \sigma(X), \dots, \sigma^n(X)]$, assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

GEN Flxq_autsum(GEN a, long n, GEN T, ulong p) see FpXQ_autsum.

GEN Flxq_auttrace(GEN a, ulong n, GEN T, ulong p) see FpXQ_auttrace.

GEN Flxq_auttrace_pre(GEN a, ulong n, GEN T, ulong p, ulong pi)

GEN Flxq_ffisom_inv(GEN S, GEN T, ulong p), as FpXQ_ffisom_inv.

GEN Flx_Flxq_eval(GEN f, GEN x, GEN T, ulong p) returns f(x).

GEN Flx_Flxq_eval_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)

GEN Flx_FlxqV_eval(GEN f, GEN x, GEN T, ulong p), see FpX_FpXQV_eval.

GEN Flx_FlxqV_eval_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)

GEN FlxC_Flxq_eval(GEN C, GEN x, GEN T, ulong p), see FpXC_FpXQ_eval.

GEN FlxC_Flxq_eval_pre(GEN C, GEN x, GEN T, ulong p, ulong pi)

GEN FlxC_FlxqV_eval(GEN C, GEN V, GEN T, ulong p) see FpXC_FpXQV_eval.

GEN FlxC_FlxqV_eval_pre(GEN C, GEN V, GEN T, ulong p, ulong pi)

GEN FlxqV_roots_to_pol(GEN V, GEN T, ulong p, long v) as FqV_roots_to_pol returning an FlxqX in variable v.

int Flxq_issquare(GEN x, GEN T, ulong p) returns 1 if x is a square and 0 otherwise. Assume that T is irreducible mod p.

int Flxq_is2npower(GEN x, long n, GEN T, ulong p) returns 1 if x is a 2^n-th power and 0 otherwise. Assume that T is irreducible mod p.

GEN Flxq_order(GEN a, GEN ord, GEN T, ulong p) as FpXQ_order.

GEN Flxq_log(GEN a, GEN g, GEN ord, GEN T, ulong p) as FpXQ_log

GEN Flxq_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN *zn) as FpXQ_sqrtn.

GEN Flxq_sqrt(GEN x, GEN T, ulong p) returns a square root of x. Return NULL if x is not a square.

GEN Flxq_sqrt_pre(GEN x, GEN T, ulong p, ulong pi)

`GEN Flxq_lroot(GEN a, GEN T, ulong p)` returns x such that $x^p = a$.

`GEN Flxq_lroot_pre(GEN a, GEN T, ulong p, ulong pi)`

`GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p)` assuming that `V=Flxq_powers(s,p-1,T,p)` where $s(x)^p \equiv x \pmod{T(x), p}$, returns b such that $b^p = a$. Only useful if p is less than the degree of T.

`GEN Flxq_lroot_fast_pre(GEN a, GEN V, GEN T, ulong p, ulong pi)`

`GEN Flxq_charpoly(GEN x, GEN T, ulong p)` returns the characteristic polynomial of `x`

`GEN Flxq_minpoly(GEN x, GEN T, ulong p)` returns the minimal polynomial of `x`

`GEN Flxq_minpoly_pre(GEN x, GEN T, ulong p, ulong pi)`

`ulong Flxq_norm(GEN x, GEN T, ulong p)` returns the norm of `x`

`ulong Flxq_trace(GEN x, GEN T, ulong p)` returns the trace of `x`

`GEN Flxq_conjvec(GEN x, GEN T, ulong p)` returns the conjugates $[x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}]$ where n is the degree of T.

`GEN gener_Flxq(GEN T, ulong p, GEN *po)` returns a primitive root modulo (T, p). T is an `Flx` assumed to be irreducible modulo the prime p. If `po` is not `NULL` it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

7.3.23 `FlxX`. See `FpXX` operations. In this section, we assume pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN pol1_FlxX(long vX, long sx)` returns the unit `FlxX` as a `t_POL` in variable `vX` which only coefficient is `pol1_Flx(sx)`.

`GEN polx_FlxX(long vX, long sx)` returns the variable X as a degree 1 `t_POL` with `Flx` coefficients in the variable x.

`long FlxY_degreex(GEN P)` return the degree of P with respect to the secondary variable.

`GEN FlxX_add(GEN P, GEN Q, ulong p)`

`GEN FlxX_sub(GEN P, GEN Q, ulong p)`

`GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)`

`GEN FlxX_double(GEN x, ulong p)`

`GEN FlxX_triple(GEN x, ulong p)`

`GEN FlxX_neg(GEN x, ulong p)`

`GEN FlxY_Flx_add(GEN x, GEN y, ulong p)`

`GEN FlxY_Flx_sub(GEN x, GEN y, ulong p)`

`GEN FlxY_Flx_mul(GEN x, GEN y, ulong p)`

`GEN FlxY_Flx_mul_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN FlxY_Flx_div(GEN x, GEN y, ulong p)` divides the coefficients of x by y using `Flx_div`.

`GEN FlxY_Flx_div_pre(GEN x, GEN y, ulong p, ulong pi)`

GEN FlxX_deriv(GEN P, ulong p) returns the derivative of P with respect to the main variable.

GEN FlxX_Laplace(GEN x, ulong p)

GEN FlxX_invLaplace(GEN x, ulong p)

GEN FlxY_evalx(GEN P, ulong z, ulong p) P being an FlxY, returns the Flx $P(z, Y)$, where Y is the main variable of P.

GEN FlxY_evalx_pre(GEN P, ulong z, ulong p, ulong pi)

GEN FlxX_translate1(GEN P, ulong p, long n) P being an FlxX with all coefficients of degree at most n, return $(P(x, Y + 1)$, where Y is the main variable of P.

GEN zlxX_translate1(GEN P, ulong p, long e, long n) P being an zlxX with all coefficients of degree at most n, return $(P(x, Y + 1)$ modulo p^e for prime p , where Y is the main variable of P.

GEN FlxY_Flx_translate(GEN P, GEN f, ulong p) P being an FlxY and f being an Flx, return $(P(x, Y + f(x))$, where Y is the main variable of P.

ulong FlxY_evalx_powers_pre(GEN P, GEN xp, ulong p, ulong pi), xp being the vector $[1, x, \ldots, x^n]$, where n is larger or equal to the degree of P in X, return $P(x, Y)$, where Y is the main variable of Q.

ulong FlxY_eval_powers_pre(GEN P, GEN xp, GEN yp, ulong p, ulong pi), xp being the vector $[1, x, \ldots, x^n]$, where n is larger or equal to the degree of P in X and yp being the vector $[1, y, \ldots, y^m]$, where m is larger or equal to the degree of P in Y return $P(x, y)$.

GEN FlxY_Flxq_evalx(GEN x, GEN y, GEN T, ulong p) as FpXY_FpXQ_evalx.

GEN FlxY_Flxq_evalx_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxY_FlxqV_evalx(GEN x, GEN V, GEN T, ulong p) as FpXY_FpXQV_evalx.

GEN FlxY_FlxqV_evalx_pre(GEN x, GEN V, GEN T, ulong p, ulong pi)

GEN FlxX_renormalize(GEN x, long l), as normalizepol, where l = lg(x), in place.

GEN FlxX_resultant(GEN u, GEN v, ulong p, long sv) Returns $\mathrm{Res}_X(u, v)$, which is an Flx. The coefficients of u and v are assumed to be in the variable v.

GEN FlxX_resultant_pre(GEN u, GEN v, ulong p, ulong pi, long sv)

GEN Flx_FlxY_resultant(GEN a, GEN b, ulong p) Returns $\mathrm{Res}_x(a, b)$, which is an Flx in the main variable of b.

GEN FlxX_blocks(GEN P, long n, long m, long sv), as RgX_blocks, where v is the secondary variable.

GEN FlxX_shift(GEN a, long n, long sv), as RgX_shift_shallow, where v is the secondary variable.

GEN FlxX_swap(GEN x, long n, long ws), as RgXY_swap.

GEN FlxYqq_pow(GEN x, GEN n, GEN S, GEN T, ulong p), as FpXYQQ_pow.

7.3.24 FlxXV, FlxXC, FlxXM. See FpXX operations.

GEN FlxXC_sub(GEN x, GEN y, ulong p)

7.3.25 `FlxqX`. See `FpXQX` operations.

7.3.25.1 Preconditioned reduction.

For faster reduction, the modulus `S` can be replaced by an extended modulus, which is an `FlxqXT`, in all `FlxqXQ`-classes functions, and in `FlxqX_rem` and `FlxqX_divrem`.

`GEN FlxqX_get_red(GEN S, GEN T, ulong p)` returns the extended modulus `eS`.

`GEN FlxqX_get_red_pre(GEN S, GEN T, ulong p, ulong pi)`, where pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FlxqX_mod(GEN eS)` returns the underlying modulus `S`.

`GEN get_FlxqX_var(GEN eS)` returns the variable number of the modulus.

`GEN get_FlxqX_degree(GEN eS)` returns the degree of the modulus.

7.3.25.2 basic functions.

In this section, pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN random_FlxqX(long d, long v, GEN T, ulong p)` returns a random `FlxqX` in variable `v`, of degree less than `d`.

`GEN zxX_to_Kronecker(GEN P, GEN Q)` assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P.

`GEN Kronecker_to_FlxqX(GEN z, GEN T, ulong p)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t)$ mod $(p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN Kronecker_to_FlxqX_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_red(GEN z, GEN T, ulong p)`

`GEN FlxqX_red_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_normalize(GEN z, GEN T, ulong p)`

`GEN FlxqX_normalize_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN FlxqX_mul_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN FlxqX_Flxq_mul(GEN P, GEN U, GEN T, ulong p)`

`GEN FlxqX_Flxq_mul_pre(GEN P, GEN U, GEN T, ulong p, ulong pi)`

`GEN FlxqX_Flxq_mul_to_monic(GEN P, GEN U, GEN T, ulong p)` returns $P * U$ assuming the result is monic of the same degree as P (in particular $U \neq 0$).

`GEN FlxqX_Flxq_mul_to_monic_pre(GEN P, GEN U, GEN T, ulong p, ulong pi)`

`GEN FlxqX_sqr(GEN x, GEN T, ulong p)`

GEN FlxqX_sqr_pre(GEN x, GEN T, ulong p, ulong pi)

GEN FlxqX_powu(GEN x, ulong n, GEN T, ulong p)

GEN FlxqX_powu_pre(GEN x, ulong n, GEN T, ulong p, ulong pi)

GEN FlxqX_divrem(GEN x, GEN y, GEN T, ulong p, GEN *pr)

GEN FlxqX_divrem_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN *pr)

GEN FlxqX_div(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX_div_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_div_by_X_x(GEN a, GEN x, GEN T, ulong p, GEN *r)

GEN FlxqX_div_by_X_x_pre(GEN a, GEN x, GEN T, ulong p, ulong pi, GEN *r)

GEN FlxqX_rem(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX_rem_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_invBarrett(GEN T, GEN Q, ulong p)

GEN FlxqX_invBarrett_pre(GEN T, GEN Q, ulong p, ulong pi)

GEN FlxqX_gcd(GEN x, GEN y, ulong p) returns a (not necessarily monic) greatest common divisor of x and y.

GEN FlxqX_gcd_pre(GEN x, GEN y, ulong p, ulong pi)

GEN FlxqX_extgcd(GEN x, GEN y, GEN T, ulong p, GEN *ptu, GEN *ptv)

GEN FlxqX_extgcd_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN *ptu, GEN *ptv)

GEN FlxqX_halfgcd(GEN x, GEN y, GEN T, ulong p), see FpX_halfgcd.

GEN FlxqX_halfgcd_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_halfgcd_all(GEN x, GEN y, GEN T, ulong p, GEN *a, GEN *b), see FpX_halfgcd_all.

GEN FlxqX_halfgcd_all_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN *a, GEN *b), see FpX_halfgcd_all_pre.

GEN FlxqX_resultant(GEN x, GEN y, GEN T, ulong p), see FpX_resultant.

GEN FlxqX_resultant_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_saferesultant(GEN P, GEN Q, GEN T, ulong p) Returns the resultant of P and Q if Euclid's algorithm succeeds and NULL otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if noninvertible leading terms occur).

GEN FlxqX_composedsum(GEN P, GEN Q, GEN T, ulong p)

GEN FlxqX_disc(GEN x, GEN T, ulong p)

GEN FlxqX_eval(GEN x, GEN y, GEN T, ulong p) evaluates the FlxqX x at the Flxq y. The result is an Flxq.

GEN FlxqXV_prod(GEN V, GEN T, ulong p)

`GEN FlxqX_safegcd(GEN P, GEN Q, GEN T, ulong p)` Returns the *monic* GCD of P and Q if Euclid's algorithm succeeds and `NULL` otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if noninvertible leading terms occur).

`GEN FlxqX_dotproduct(GEN x, GEN y, GEN T, ulong p)` returns the scalar product of the coefficients of x and y.

`GEN FlxqX_Newton(GEN x, long n, GEN T, ulong p)`

`GEN FlxqX_Newton_pre(GEN x, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqX_fromNewton(GEN x, GEN T, ulong p)`

`GEN FlxqX_fromNewton_pre(GEN x, GEN T, ulong p, ulong pi)` We assume pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`long FlxqX_is_squarefree(GEN S, GEN T, ulong p)`, as `FpX_is_squarefree`.

`long FlxqX_ispower(GEN f, ulong k, GEN T, ulong p, GEN *pt)` return 1 if the `FlxqX` f is a k-th power, 0 otherwise. If `pt` is not `NULL`, set it to g such that $g^k = f$.

`GEN FlxqX_Frobenius(GEN S, GEN T, ulong p)`, as `FpXQX_Frobenius`

`GEN FlxqX_Frobenius_pre(GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqX_roots(GEN f, GEN T, ulong p)` return the roots of `f` in $\mathbf{F}_p[X]/(T)$. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FlxqX_factor(GEN f, GEN T, ulong p)` return the factorization of `f` over $\mathbf{F}_p[X]/(T)$. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FlxqX_factor_squarefree(GEN f, GEN T, ulong p)` returns the squarefree factorization of f, see `FpX_factor_squarefree`.

`GEN FlxqX_factor_squarefree_pre(GEN f, GEN T, ulong p, ulong pi)`

`GEN FlxqX_ddf(GEN f, GEN T, ulong p)` as `FpX_ddf`.

`long FlxqX_ddf_degree(GEN f, GEN XP, GEN T, GEN p)`, as `FpX_ddf_degree`.

`GEN FlxqX_degfact(GEN f, GEN T, ulong p)`, as `FpX_degfact`.

`long FlxqX_nbroots(GEN S, GEN T, ulong p)`, as `FpX_nbroots`.

`long FlxqX_nbfact(GEN S, GEN T, ulong p)`, as `FpX_nbfact`.

`long FlxqX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, ulong p)`, as `FpX_nbfact_Frobenius`.

`GEN FlxqX_nbfact_by_degree(GEN z, long *nb, GEN T, ulong p)` Assume that the `FlxqX` z is squarefree mod the prime p. Returns a `t_VECSMALL` D with $\deg z$ entries, such that $D[i]$ is the number of irreducible factors of degree i. Set `nb` to the total number of irreducible factors (the sum of the $D[i]$).

`GEN FlxqX_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p)` as `FpX_FpXQ_eval`.

`GEN FlxqX_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqX_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p)` as `FpX_FpXQV_eval`.

`GEN FlxqX_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)`

GEN FlxqXC_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p) as FpXC_FpXQ_eval.

GEN FlxqXC_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXC_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p) as FpXC_FpXQV_eval.

GEN FlxqXC_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)

7.3.26 FlxqXQ. See FpXQXQ operations. In this section, pi is a pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN FlxqXQ_mul(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ_mul_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_sqr(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_sqr_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_inv(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_inv_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_invsafe_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_div(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ_div_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_pow(GEN x, GEN n, GEN S, GEN T, ulong p)

GEN FlxqXQ_pow_pre(GEN x, GEN n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_powu(GEN x, ulong n, GEN S, GEN T, ulong p)

GEN FlxqXQ_powu_pre(GEN x, ulong n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_powers(GEN x, long n, GEN S, GEN T, ulong p)

GEN FlxqXQ_powers_pre(GEN x, long n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_matrix_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)

GEN FlxqXQ_autpow(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autpow

GEN FlxqXQ_autpow_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_autsum(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autsum

GEN FlxqXQ_autsum_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_auttrace(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_auttrace

GEN FlxqXQ_auttrace_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXQ_halfFrobenius(GEN A, GEN S, GEN T, ulong p), as FpXQXQ_halfFrobenius

GEN FlxqXQ_minpoly(GEN x, GEN S, GEN T, ulong p), as FpXQ_minpoly

GEN FlxqXQ_minpoly_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)

7.3.27 `FlxqXn`. See `FpXn` operations. In this section, we assume pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN FlxXn_red(GEN a, long n)` returns a modulo X^n.

`GEN FlxqXn_mul(GEN a, GEN b, long n, GEN T, ulong p)`

`GEN FlxqXn_mul_pre(GEN a, GEN b, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_sqr(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_sqr_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_inv(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_inv_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_div(GEN a, GEN b, long n, GEN T, ulong p)`

`GEN FlxqXn_div_pre(GEN a, GEN b, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_expint(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_expint_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

7.3.28 `F2x`. An `F2x z` is a `t_VECSMALL` representing a polynomial over $\mathbf{F}_2[X]$. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn(`v`)` for some variable v and the coefficients are given by the bits of remaining words by increasing degree.

7.3.28.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus (`FlxT`) in all `Flxq`-classes functions, and in `Flx_divrem`.

`GEN F2x_get_red(GEN T)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_F2x_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_F2x_var(GEN eT)` returns the variable number of the modulus.

`GEN get_F2x_degree(GEN eT)` returns the degree of the modulus.

7.3.28.2 Basic operations.

`ulong F2x_coeff(GEN x, long i)` returns the coefficient $i \geq 0$ of x.

`void F2x_clear(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 0.

`void F2x_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 0$ of x.

`void F2x_set(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 1.

`GEN F2x_copy(GEN x)`

`GEN Flx_to_F2x(GEN x)`

`GEN Z_to_F2x(GEN x, long sv)`

`GEN ZX_to_F2x(GEN x)`

`GEN F2v_to_F2x(GEN x, long sv)`

`GEN F2x_to_Flx(GEN x)`

`GEN F2x_to_F2xX(GEN x, long sv)`

`GEN F2x_to_ZX(GEN x)`

`GEN pol0_F2x(long sv)` returns a zero `F2x` in variable v.

`GEN zero_F2x(long sv)` alias for `pol0_F2x`.

`GEN pol1_F2x(long sv)` returns the `F2x` in variable v constant to 1.

`GEN polx_F2x(long sv)` returns the variable v as degree 1 `F2x`.

`GEN monomial_F2x(long d, long sv)` returns the `F2x` X^d in variable v.

`GEN random_F2x(long d, long sv)` returns a random `F2x` in variable `v`, of degree less than `d`.

`long F2x_degree(GEN x)` returns the degree of the `F2x x`. The degree of 0 is defined as -1.

`GEN F2x_recip(GEN x)`

`int F2x_equal1(GEN x)`

`int F2x_equal(GEN x, GEN y)`

`GEN F2x_1_add(GEN y)` returns `y+1` where `y` is a `Flx`.

`GEN F2x_add(GEN x, GEN y)`

`GEN F2x_mul(GEN x, GEN y)`

`GEN F2x_sqr(GEN x)`

`GEN F2x_divrem(GEN x, GEN y, GEN *pr)`

`GEN F2x_rem(GEN x, GEN y)`

`GEN F2x_div(GEN x, GEN y)`

`GEN F2x_renormalize(GEN x, long lx)`

`GEN F2x_deriv(GEN x)`

`GEN F2x_deflate(GEN x, long d)`

`ulong F2x_eval(GEN P, ulong u)` returns $P(u)$.

`void F2x_shift(GEN x, long d)` as `RgX_shift`

`void F2x_even_odd(GEN P, GEN *pe, GEN *po)` as `RgX_even_odd`

`long F2x_valrem(GEN x, GEN *Z)`

`GEN F2x_extgcd(GEN a, GEN b, GEN *ptu, GEN *ptv)`

`GEN F2x_gcd(GEN a, GEN b)`

`GEN F2x_halfgcd(GEN a, GEN b)`

`int F2x_issquare(GEN x)` returns 1 if x is a square of a `F2x` and 0 otherwise.

`int F2x_is_irred(GEN f)`, as `FpX_is_irred`.

`GEN F2x_degfact(GEN f)` as `FpX_degfact`.

`GEN F2x_sqrt(GEN x)` returns the squareroot of x, assuming x is a square of a `F2x`.

`GEN F2x_Frobenius(GEN T)`

`GEN F2x_matFrobenius(GEN T)`

`GEN F2x_factor(GEN f)`

`GEN F2x_factor_squarefree(GEN f)`

`GEN F2x_ddf(GEN f)`

`GEN F2x_Teichmuller(GEN P, long n)` Return a `ZX` Q such that $P \equiv Q \pmod 2$ and $Q(X^p) = 0 \pmod{Q, 2^n}$.

7.3.29 `F2xq`. See `FpXQ` operations.

`GEN F2xq_mul(GEN x, GEN y, GEN T)`

`GEN F2xq_sqr(GEN x, GEN T)`

`GEN F2xq_div(GEN x, GEN y, GEN T)`

`GEN F2xq_inv(GEN x, GEN T)`

`GEN F2xq_invsafe(GEN x, GEN T)`

`GEN F2xq_pow(GEN x, GEN n, GEN T)`

`GEN F2xq_powu(GEN x, ulong n, GEN T)`

`GEN F2xq_pow_init(GEN x, GEN n, long k, GEN T)`

`GEN F2xq_pow_table(GEN R, GEN n, GEN T)`

`ulong F2xq_trace(GEN x, GEN T)`

`GEN F2xq_conjvec(GEN x, GEN T)` returns the vector of conjugates $[x, x^2, x^{2^2}, \ldots, x^{2^{n-1}}]$ where n is the degree of T.

`GEN F2xq_log(GEN a, GEN g, GEN ord, GEN T)`

`GEN F2xq_order(GEN a, GEN ord, GEN T)`

`GEN F2xq_Artin_Schreier(GEN a, GEN T)` returns a solution of $x^2 + x = a$, assuming it exists.

`GEN F2xq_sqrt(GEN a, GEN T)`

`GEN F2xq_sqrt_fast(GEN a, GEN s, GEN T)` assuming that $s^2 \equiv x \pmod{T(x)}$, computes $b \equiv a(s) \pmod T$ so that $b^2 = a$.

`GEN F2xq_sqrtn(GEN a, GEN n, GEN T, GEN *zeta)`

`GEN gener_F2xq(GEN T, GEN *po)`

`GEN F2xq_powers(GEN x, long n, GEN T)`

`GEN F2xq_matrix_pow(GEN x, long m, long n, GEN T)`

`GEN F2x_F2xq_eval(GEN f, GEN x, GEN T)`

`GEN F2x_F2xqV_eval(GEN f, GEN x, GEN T)`, see `FpX_FpXQV_eval`.

`GEN F2xq_autpow(GEN a, long n, GEN T)` computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_2[X]/T(X)$.

`GEN F2xqV_roots_to_pol(GEN V, GEN T, long v)` as `FqV_roots_to_pol` returning an `F2xqX` in variable v.

7.3.30 `F2xn`. See `FpXn` operations.

`GEN F2xn_red(GEN a, long n)`

`GEN F2xn_div(GEN x, GEN y, long e)`

`GEN F2xn_inv(GEN x, long e)`

7.3.31 `F2xqV`, `F2xqM`.. See `FqV`, `FqM` operations.

`GEN F2xqM_F2xqC_gauss(GEN a, GEN b, GEN T)`

`GEN F2xqM_F2xqC_invimage(GEN a, GEN b, GEN T)`

`GEN F2xqM_F2xqC_mul(GEN a, GEN b, GEN T)`

`GEN F2xqM_deplin(GEN x, GEN T)`

`GEN F2xqM_det(GEN a, GEN T)`

`GEN F2xqM_gauss(GEN a, GEN b, GEN T)`

`GEN F2xqM_image(GEN x, GEN T)`

`GEN F2xqM_indexrank(GEN x, GEN T)`

`GEN F2xqM_inv(GEN a, GEN T)`

`GEN F2xqM_invimage(GEN a, GEN b, GEN T)`

`GEN F2xqM_ker(GEN x, GEN T)`

`GEN F2xqM_mul(GEN a, GEN b, GEN T)`

`long F2xqM_rank(GEN x, GEN T)`

`GEN F2xqM_suppl(GEN x, GEN T)`

`GEN matid_F2xqM(long n, GEN T)`

7.3.32 `F2xX`.. See `FpXX` operations.

`GEN ZXX_to_F2xX(GEN x, long v)`

`GEN FlxX_to_F2xX(GEN x)`

`GEN F2xX_to_FlxX(GEN B)`

`GEN F2xX_to_F2xC(GEN B, long N, long sv)`

`GEN F2xXV_to_F2xM(GEN B, long N, long sv)`

`GEN F2xX_to_ZXX(GEN B)`

`GEN F2xX_renormalize(GEN x, long lx)`

`GEN F2xX_shift(GEN a, long n, long sv)`, as `RgX_shift_shallow`, where v is the secondary variable.

`long F2xY_degreex(GEN P)` return the degree of P with respect to the secondary variable.

`GEN pol1_F2xX(long v, long sv)`

`GEN polx_F2xX(long v, long sv)`

`GEN F2xX_add(GEN x, GEN y)`

`GEN F2xX_F2x_add(GEN x, GEN y)`

`GEN F2xX_F2x_mul(GEN x, GEN y)`

`GEN F2xX_deriv(GEN P)` returns the derivative of `P` with respect to the main variable.

`GEN Kronecker_to_F2xqX(GEN z, GEN T)`

`GEN F2xX_to_Kronecker(GEN z, GEN T)`

`GEN F2xY_F2xq_evalx(GEN x, GEN y, GEN T)` as `FpXY_FpXQ_evalx`.

`GEN F2xY_F2xqV_evalx(GEN x, GEN V, GEN T)` as `FpXY_FpXQV_evalx`.

7.3.33 `F2xXV/F2xXC`.. See `FpXXV` operations.

`GEN FlxXC_to_F2xXC(GEN B)`

`GEN F2xXC_to_ZXXC(GEN B)`

7.3.34 `F2xqX`.. See `FlxqX` operations.

7.3.34.1 Preconditioned reduction.

For faster reduction, the modulus `S` can be replaced by an extended modulus, which is an `F2xqXT`, in all `F2xqXQ`-classes functions, and in `F2xqX_rem` and `F2xqX_divrem`.

`GEN F2xqX_get_red(GEN S, GEN T)` returns the extended modulus `eS`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_F2xqX_mod(GEN eS)` returns the underlying modulus `S`.

`GEN get_F2xqX_var(GEN eS)` returns the variable number of the modulus.

`GEN get_F2xqX_degree(GEN eS)` returns the degree of the modulus.

7.3.34.2 basic functions.

`GEN random_F2xqX(long d, long v, GEN T, ulong p)` returns a random `F2xqX` in variable `v`, of degree less than `d`.

`GEN F2xqX_red(GEN z, GEN T)`

`GEN F2xqX_normalize(GEN z, GEN T)`

`GEN F2xqX_F2xq_mul(GEN P, GEN U, GEN T)`

`GEN F2xqX_F2xq_mul_to_monic(GEN P, GEN U, GEN T)`

`GEN F2xqX_mul(GEN x, GEN y, GEN T)`

`GEN F2xqX_sqr(GEN x, GEN T)`

`GEN F2xqX_powu(GEN x, ulong n, GEN T)`

`GEN F2xqX_rem(GEN x, GEN y, GEN T)`

`GEN F2xqX_div(GEN x, GEN y, GEN T)`

`GEN F2xqX_divrem(GEN x, GEN y, GEN T, GEN *pr)`

`GEN F2xqXQ_inv(GEN x, GEN S, GEN T)`

`GEN F2xqXQ_invsafe(GEN x, GEN S, GEN T)`

`GEN F2xqX_invBarrett(GEN T, GEN Q)`

`GEN F2xqX_extgcd(GEN x, GEN y, GEN T, GEN *ptu, GEN *ptv)`

`GEN F2xqX_gcd(GEN x, GEN y, GEN T)`

`GEN F2xqX_halfgcd(GEN x, GEN y, GEN T)`

`GEN F2xqX_halfgcd_all(GEN x, GEN y, GEN T, GEN *a, GEN *b)`

`GEN F2xqX_resultant(GEN x, GEN y, GEN T)`

`GEN F2xqX_disc(GEN x, GEN T)`

`GEN F2xqXV_prod(GEN V, GEN T)`

`long F2xqX_ispower(GEN f, ulong k, GEN T, GEN *pt)`

`GEN F2xqX_F2xqXQ_eval(GEN Q, GEN x, GEN S, GEN T)` as `FpX_FpXQ_eval`.

`GEN F2xqX_F2xqXQV_eval(GEN P, GEN V, GEN S, GEN T)` as `FpX_FpXQV_eval`.

`GEN F2xqX_Frobenius(GEN S, GEN T)`, as `FpXQX_Frobenius`

`GEN F2xqX_roots(GEN f, GEN T)` return the roots of `f` in $\mathbf{F}_2[X]/(T)$. Assumes `T` irreducible in $\mathbf{F}_2[X]$.

`GEN F2xqX_factor(GEN f, GEN T)` return the factorization of `f` over $\mathbf{F}_2[X]/(T)$. Assumes `T` irreducible in $\mathbf{F}_2[X]$.

`GEN F2xqX_factor_squarefree(GEN f, GEN T)` as `FlxqX_factor_squarefree`.

`GEN F2xqX_ddf(GEN f, GEN T)` as `FpX_ddf`.

`GEN F2xqX_degfact(GEN f, GEN T)` as `FpX_degfact`.

`long F2xqX_nbroots(GEN f, GEN T)` as `FpX_nbroots`.

7.3.35 `F2xqXQ`.. See `FlxqXQ` operations.

`GEN FlxqXQ_inv(GEN x, GEN S, GEN T)`

`GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T)`

`GEN F2xqXQ_mul(GEN x, GEN y, GEN S, GEN T)`

`GEN F2xqXQ_sqr(GEN x, GEN S, GEN T)`

`GEN F2xqXQ_pow(GEN x, GEN n, GEN S, GEN T)`

`GEN F2xqXQ_powers(GEN x, long n, GEN S, GEN T)`

`GEN F2xqXQ_autpow(GEN a, long n, GEN S, GEN T)` as `FpXQXQ_autpow`

`GEN F2xqXQ_auttrace(GEN a, long n, GEN S, GEN T)`. Let σ be the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$ and $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$; returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[3]$.

`GEN F2xqXQV_red(GEN x, GEN S, GEN T)`

7.3.36 Functions returning objects with `t_INTMOD` coefficients.

Those functions are mostly needed for interface reasons: `t_INTMOD`s should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

`GEN Fp_to_mod(GEN z, GEN p)`, `z` a `t_INT`. Returns `z * Mod(1,p)`, normalized. Hence the returned value is a `t_INTMOD`.

`GEN FpX_to_mod(GEN z, GEN p)`, `z` a `ZX`. Returns `z * Mod(1,p)`, normalized. Hence the returned value has `t_INTMOD` coefficients.

`GEN FpC_to_mod(GEN z, GEN p)`, `z` a `ZC`. Returns `Col(z) * Mod(1,p)`, a `t_COL` with `t_INTMOD` coefficients.

`GEN FpV_to_mod(GEN z, GEN p)`, `z` a `ZV`. Returns `Vec(z) * Mod(1,p)`, a `t_VEC` with `t_INTMOD` coefficients.

`GEN FpVV_to_mod(GEN z, GEN p)`, `z` a `ZVV`. Returns `Vec(z) * Mod(1,p)`, a `t_VEC` of `t_VEC` with `t_INTMOD` coefficients.

`GEN FpM_to_mod(GEN z, GEN p)`, `z` a `ZM`. Returns `z * Mod(1,p)`, with `t_INTMOD` coefficients.

`GEN F2c_to_mod(GEN x)`

`GEN F3c_to_mod(GEN x)`

`GEN F2m_to_mod(GEN x)`

`GEN F3m_to_mod(GEN x)`

`GEN Flc_to_mod(GEN z)`

`GEN Flm_to_mod(GEN z)`

`GEN FqC_to_mod(GEN z, GEN T, GEN p)`

`GEN FqM_to_mod(GEN z, GEN T, GEN p)`

`GEN FpXC_to_mod(GEN V, GEN p)`

`GEN FpXM_to_mod(GEN V, GEN p)`

`GEN FpXQC_to_mod(GEN V, GEN T, GEN p)` V being a vector of `FpXQ`, converts each entry to a `t_POLMOD` with `t_INTMOD` coefficients, and return a `t_COL`.

`GEN FpXQX_to_mod(GEN P, GEN T, GEN p)` P being a `FpXQX`, converts each coefficient to a `t_POLMOD` with `t_INTMOD` coefficients.

`GEN FqX_to_mod(GEN P, GEN T, GEN p)` same but allow `T` = `NULL`.

`GEN FqXC_to_mod(GEN P, GEN T, GEN p)`

`GEN FqXM_to_mod(GEN P, GEN T, GEN p)`

`GEN QXQ_to_mod_shallow(GEN x, GEN T)` x a `QXQ`, which is a lifted representative of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Convert it to a `t_POLMOD` modulo T; no reduction mod T is attempted: the representatives should be already reduced. Shallow function.

`GEN QXQV_to_mod(GEN V, GEN T)` V a vector of `QXQ`, which are lifted representatives of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Return a vector where all nonrational entries are converted to `t_POLMOD` modulo T; no reduction mod T is attempted: the representatives should be already reduced. Used to normalize the output of `nfroots`.

`GEN QXQX_to_mod_shallow(GEN P, GEN T)` P a polynomial with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

`GEN QXQC_to_mod_shallow(GEN V, GEN T)` V a vector with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

`GEN QXQM_to_mod_shallow(GEN M, GEN T)` M a matrix with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

`GEN QXQXV_to_mod(GEN V, GEN T)` V a vector of polynomials whose coefficients are `QXQ`. Analogous to `QXQV_to_mod`. Used to normalize the output of `nffactor`.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply a conversion function to map them to a finite field, a function from the modular kernel, then `*_to_mod`:

`GEN rootmod(GEN f, GEN p)`, applies `FpX_roots`.

`GEN rootmod2(GEN f, GEN p)`, (now) identical to `rootmod`.

`GEN rootmod0(GEN f, GEN p, long flag)`, (now) identical to `rootmod`; ignores *flag*.

`GEN factmod(GEN f, GEN p)` applies `*_factor`.

`GEN simplefactmod(GEN f, GEN p)` applies `*_degfact`.

7.3.37 Slow Chinese remainder theorem over Z. The routines in this section have quadratic time complexity with respect to the input size; see the routines in the next two sections for quasi-linear time variants.

`GEN Z_chinese(GEN a, GEN b, GEN A, GEN B)` returns the integer in $[0, \mathrm{lcm}(A,B)[$ congruent to a mod A and b mod B, assuming it exists; in other words, that a and b are congruent mod $\gcd(A,B)$.

`GEN Z_chinese_all(GEN a, GEN b, GEN A, GEN B, GEN *pC)` as `Z_chinese`, setting `*pC` to the lcm of A and B.

`GEN Z_chinese_coprime(GEN a, GEN b, GEN A, GEN B, GEN C)`, as `Z_chinese`, assuming that $\gcd(A,B) = 1$ and that $C = \mathrm{lcm}(A,B) = AB$.

`ulong u_chinese_coprime(ulong a, ulong b, ulong A, ulong B, ulong C)`, as `Z_chinese_coprime` for `ulong` inputs and output.

`void Z_chinese_pre(GEN A, GEN B, GEN *pC, GEN *pU, GEN *pd)` initializes chinese remainder computations modulo A and B. Sets `*pC` to $\mathrm{lcm}(A,B)$, `*pd` to $\gcd(A,B)$, `*pU` to an integer congruent to 0 mod (A/d) and 1 mod (B/d). It is allowed to set `pd = NULL`, in which case, d is still computed, but not saved.

`GEN Z_chinese_post(GEN a, GEN b, GEN C, GEN U, GEN d)` returns the solution to the chinese remainder problem x congruent to a mod A and b mod B, where C, U, d were set in `Z_chinese_pre`. If d is `NULL`, assume the problem has a solution. Otherwise, return `NULL` if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer H which satisfies $|H| < B$ for some known bound B; we are given pairs (H_p, p) with H congruent to H_p mod p and all p pairwise coprime.

Given `H` congruent to H_p modulo a number of p, whose product is q, and a new pair (`Hp`, `p`), `p` coprime to q, the following incremental functions use the chinese remainder theorem (CRT) to find a new `H`, congruent to the preceding one modulo q, but also to `Hp` modulo `p`. It is defined uniquely modulo qp, and we choose the centered representative. When P is larger than $2B$, we have `H` $= H$, but of course, the value of `H` may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl` `Hp` in $[0, p-1]$, returns the centered representative `H` congruent to `Hp` modulo `p`.

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT` `*H`, centered modulo `*q`, a new pair (`Hp`, `p`) with `p` coprime to `q`, this function updates `*H` so that it also becomes congruent to (`Hp`, `p`), and `*q` to the product`qp` = `p` $\cdot$ `*q`. It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation: v is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over **Z**, moduli not assumed coprime).

As above, for H a `ZM`: we assume that H and all `Hp` have dimension > 0. The original `*H` is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for H a `ZX`: note that the degree may increase or decrease. The original `*H` is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above, for H a matrix whose coefficient are `ZX`. The original `*H` is destroyed. The entries of H are not normalized, use `ZX_renormalize` for this.

`GEN ZXM_init_CRT(GEN Hp, long deg, ulong p)` where `deg` is the maximal degree of all the `Hp`

`int ZXM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

7.3.38 Fast remainders.

The routines in these section are asymptotically fast (quasi-linear time in the input size).

`GEN Z_ZV_mod(GEN A, GEN P)` given a `t_INT` A and a vector P of positive pairwise coprime integers of length $n \geq 1$, return a vector B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The vector P may be a `t_VEC` or a `t_VECSMALL` (treated as `ulong`s) and B has the same type as P.

`GEN Z_nv_mod(GEN A, GEN P)` given a `t_INT` A and a `t_VECSMALL` P of positive pairwise coprime integers of length $n \geq 1$, return a `t_VECSMALL` B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The entries of P and B are treated as `ulong`s.

The following low level functions allow precomputations:

`GEN ZV_producttree(GEN P)` where P is a vector of integers (or `t_VECSMALL`) of length $n \geq 1$, return the vector of `t_VEC`s $[f(P), f^2(P), \dots, f^k(P)]$ where f is the transformation $[p_1, p_2, \dots, p_m] \mapsto [p_1p_2, p_3p_4, \dots, p_{m-1}p_m]$ if m is even and $[p_1p_2, p_3p_4, \dots, p_{m-2}p_{m-1}, p_m]$ if m is odd, and $k = O(\log m)$ is minimal so that $f^k(P)$ has length 1; in other words, $f^k(P) = [p_1p_2 \dots p_m]$.

`GEN ZMV_producttree(GEN P)` where P is a vector of `ZM` of length $n \geq 1$, return the vector of `t_VEC`s $[f(P), f^2(P), \dots, f^k(P)]$ where f is the transformation $[p_1, p_2, \dots, p_m] \mapsto [p_2p_1, p_4p_3, \dots, p_mp_{m-1}]$ if m is even and $[p_2p_1, p_4p_3, \dots, p_{m-1}p_{m-2}, p_m]$ if m is odd, and $k = O(\log m)$ is minimal so that $f^k(P)$ has length 1; in other words, $f^k(P) = [p_m \dots p_2p_1]$.

`GEN Z_ZV_mod_tree(GEN A, GEN P, GEN T)` as `Z_ZV_mod` where T is the tree `ZV_producttree(P)`.

`GEN ZV_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZV` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flv` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZM_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flm` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZVV_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `t_VEC` of `t_ZV` (all of length $n \geq 1$) and P a `t_VECSMALL` of length n, the elements of P being pairwise coprime, return the vector of `FlvV` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZX_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZX` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flx` polynomials [$A \pmod{P[1]}, \ldots, A \pmod{P[n]}$], where T is the tree `ZV_producttree(P)`.

`GEN ZXC_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZXC` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxC` [$A \pmod{P[1]}, \ldots, A \pmod{P[n]}$], where T is the tree `ZV_producttree(P)`.

`GEN ZXM_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZXM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxM` [$A \pmod{P[1]}, \ldots, A \pmod{P[n]}$], where T is the tree `ZV_producttree(P)`.

`GEN ZXX_nv_mod_tree(GEN A, GEN P, GEN T, long v)` A being a `ZXX`, and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxX` [$A \pmod{P[1]}, \ldots, A \pmod{P[n]}$], where T is assumed to be the tree created by `ZV_producttree(P)`.

7.3.39 Fast Chinese remainder theorem over Z. The routines in these section are asymptotically fast (quasi-linear time in the input size) and should be used whenever the moduli are known from the start.

The simplest function is

`GEN ZV_chinese(GEN A, GEN P, GEN *pM)` let P be a vector of positive pairwise coprime integers, let A be a vector of integers of the same length $n \geq 1$ such that $0 \leq A[i] < P[i]$ for all i, and let M be the product of the elements of P. Returns the integer in $[0, M[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pM` to M. We also allow `t_VECSMALL`s for A and P (seen as vectors of unsigned integers).

`GEN ZV_chinese_center(GEN A, GEN P, GEN *pM)` As `ZV_chinese` but return integers in $[-M/2, M/2[$ instead.

The following functions allow to solve many Chinese remainder problems simultaneously, for a given set of moduli:

`GEN nxV_chinese_center(GEN A, GEN P, GEN *pt_mod)` where A is a vector of `nx` and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the `t_POL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pt_mod` is not `NULL`, set `*pt_mod` to M.

`GEN ncV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `VECSMALL`s (seen as vectors of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pt_mod` to M.

`GEN nmV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `MATSMALL`s (seen as matrices of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pM` to M. N.B.: this function uses the parallel GP interface.

`GEN nxCV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxC`s and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being

the product of the elements of P, returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \le i \le n$. If `pM` is not `NULL`, set `*pt_mod` to M.

`GEN nxMV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxM`s and P a `t_VECSMALL` of the same length $n \ge 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \le i \le n$. If `pM` is not `NULL`, set `*pM` to M. N.B.: this function uses the parallel GP interface.

The other routines allow for various precomputations :

`GEN ZV_chinesetree(GEN P, GEN T)` given P a vector of integers (or `t_VECSMALL`) and a product tree T from `ZV_producttree`(P) for the same P, return a "chinese remainder tree" R, preconditionning the solution of Chinese remainder problems modulo the $P[i]$.

`GEN ZV_chinese_tree(GEN A, GEN P, GEN T, GEN R)` return `ZV_chinese`$(A, P,$ `NULL`$)$, where T is created by `ZV_producttree`(P) and R by `ZV_chinesetree`(P, T).

`GEN ncV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R)` as `ncV_chinese_center` where T is assumed to be the tree created by `ZV_producttree(P)` and R by `ZV_chinesetree`(P, T).

`GEN nmV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R)` as `nmV_chinese_center` where T is assumed to be the tree created by `ZV_producttree(P)` and R by `ZV_chinesetree`(P, T).

`GEN nxV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R)` as `nxV_chinese_center` where T is assumed to be the tree created by `ZV_producttree(P)` and R by `ZV_chinesetree`(P, T).

`GEN nxCV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R)` as `nxCV_chinese_center` where T is assumed to be the tree created by `ZV_producttree(P)` and R by `ZV_chinesetree`(P, T).

7.3.40 Rational reconstruction.

`int Fp_ratlift(GEN x, GEN m, GEN amax, GEN bmax, GEN *a, GEN *b)`. Assuming that $0 \le x < m$, `amax` ≥ 0, and `bmax` > 0 are `t_INT`s, and that 2`amaxbmax` $< m$, attempts to recognize x as a rational a/b, i.e. to find `t_INT`s a and b such that

- $a \equiv bx$ modulo m,
- $|a| \le$ `amax`, $0 < b \le$ `bmax`,
- $\gcd(m, b) = \gcd(a, b)$.

If unsuccessful, the routine returns 0 and leaves a, b unchanged; otherwise it returns 1 and sets a and b.

In almost all applications, we actually know that a solution exists, as well as a nonzero multiple B of b, and $m = p^\ell$ is a prime power, for a prime p chosen coprime to B hence to b. Under the single assumption $\gcd(m, b) = 1$, if a solution a, b exists satisfying the three conditions above, then it is unique.

`GEN FpM_ratlift(GEN M, GEN m, GEN amax, GEN bmax, GEN denom)` given an `FpM` modulo m with reduced or `Fp_center`-ed entries, reconstructs a matrix with rational coefficients by applying `Fp_ratlift` to all entries. Assume that all preconditions for `Fp_ratlift` are satisfied, as well $\gcd(m, b) = 1$ (so that the solution is unique if it exists). Return `NULL` if the reconstruction fails, and the rational matrix otherwise. If `denom` is not `NULL` check further that all denominators divide `denom`.

`GEN FpX_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom)` as `FpM_ratlift`, where P is an `FpX`.

`GEN FpC_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom)` as `FpM_ratlift`, where P is an `FpC`.

7.3.41 Zp.

`GEN Zp_invlift(GEN b, GEN a, GEN p, long e)` let p be a prime `t_INT`, a be a `t_INT` and b a `t_INT` such that $ab \equiv 1 \bmod p$. Returns an `t_INT` A such that $A \equiv a \pmod{p}$ and $Ab \equiv 1 \bmod p^e$.

`GEN Zp_inv(GEN b, GEN p, long e)` let p be a prime `t_INT` and b be a `t_INT` Returns an `t_INT` A such that $Ab \equiv 1 \bmod p^e$.

`GEN Zp_div(GEN a, GEN b, GEN p, long e)` let p be a prime `t_INT` and a and b be a `t_INT` Returns an `t_INT` c such that $cb \equiv a \bmod p^e$.

`GEN Zp_sqrt(GEN b, GEN p, long e)` b and p being `t_INT`s, with p a prime (possibly 2), returns a `t_INT` a such that $a^2 \equiv b \bmod p^e$.

`GEN Z2_sqrt(GEN b, long e)` b being a `t_INT`s returns a `t_INT` a such that $a^2 \equiv b \bmod 2^e$.

`GEN Zp_sqrtlift(GEN b, GEN a, GEN p, long e)` let a, b, p be `t_INT`s, with $p > 2$, such that $a^2 \equiv b \bmod p$. Returns a `t_INT` A such that $A^2 \equiv b \bmod p^e$. Special case of `Zp_sqrtnlift`.

`GEN Zp_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e)` let a, b, n, p be `t_INT`s, with $n, p > 1$, and p coprime to n, such that $a^n \equiv b \bmod p$. Returns a `t_INT` A such that $A^n \equiv b \bmod p^e$. Special case of `ZpX_liftroot`.

`GEN Zp_teichmuller(GEN x, GEN p, long e, GEN pe)` for p an odd prime, x a `t_INT` coprime to p, and $pe = p^e$, returns the $(p-1)$-th root of 1 congruent to x modulo p, modulo p^e. For convenience, $p = 2$ is also allowed and we return 1 (x is 1 mod 4) or $2^e - 1$ (x is 3 mod 4).

`GEN teichmullerinit(long p, long n)` returns the values of `Zp_teichmuller` at all $x = 1, \ldots, p-1$.

`GEN Zp_exp(GEN z, GEN p, ulong e)` given a `t_INT` z (preferably reduced mod p^e), return $\exp_p(a) \bmod p^e$ (`t_INT`).

`GEN Zp_log(GEN z, GEN p, ulong e)` given a `t_INT` z (preferably reduced mod p^e), such that $a \equiv 1 \pmod{p}$, return $\log_p(a) \bmod p^e$ (`t_INT`).

7.3.42 ZpM.

`GEN ZpM_invlift(GEN M, GEN Np, GEN p, long e)` let p be a prime `t_INT`, Np be a `FpM` (modulo p) and M a `ZpM` such that $MNp \equiv 1 \bmod p$. Returns an `ZpM` N such that $N \equiv Np \pmod{p}$ and $MN \equiv 1 \bmod p^e$.

`GEN ZpM_ZpC_gauss(GEN M, GEN V, GEN q, GEN p, long e)` M be a `ZpM` (modulo $q = p^e$) such M is invertible modulo p, solve $M * X = V$ modulo p^e.

7.3.43 ZpX.

`GEN ZpX_roots(GEN f, GEN p, long e)` f a `ZX` with leading term prime to p, and without multiple roots mod p. Return a vector of `t_INT`s which are the roots of f mod p^e.

`GEN ZpX_liftroot(GEN f, GEN a, GEN p, long e)` f a `ZX` with leading term prime to p, and a a root mod p such that $v_p(f'(a)) = 0$. Return a `t_INT` which is the root of f mod p^e congruent to a mod p.

`GEN ZX_Zp_root(GEN f, GEN a, GEN p, long e)` same as `ZpX_liftroot` without the assumption $v_p(f'(a)) = 0$. Return a `t_VEC` of `t_INT`s, which are the p-adic roots of f congruent to a mod p (given modulo p^e). Assume that $0 \leq a < p$.

`GEN ZpX_liftroots(GEN f, GEN S, GEN p, long e)` f a `ZX` with leading term prime to p, and S a vector of simple roots mod p. Return a vector of `t_INT`s which are the root of f mod p^e congruent to the $S[i]$ mod p.

`GEN ZpX_liftfact(GEN A, GEN B, GEN pe, GEN p, long e)` is the routine underlying `polhensellift`. Here, p is prime defines a finite field $\mathbf{F}_p$. A is a polynomial in $\mathbf{Z}[X]$, whose leading coefficient is nonzero in $\mathbf{F}_q$. B is a vector of monic `FpX`, pairwise coprime in $\mathbf{F}_p[X]$, whose product is congruent to $A/\mathrm{lc}(A)$ in $\mathbf{F}_p[X]$. Lifts the elements of B mod `pe` $= p^e$.

`GEN ZpX_Frobenius(GEN T, GEN p, ulong e)` returns the p-adic lift of the Frobenius automorphism of $\mathbf{F}_p[X]/(T)$ to precision e.

`long ZpX_disc_val(GEN f, GEN p)` returns the valuation at p of the discriminant of f. Assume that f is a monic *separable* `ZX` and that p is a prime number. Proceeds by dynamically increasing the p-adic accuracy; infinite loop if the discriminant of f is 0.

`long ZpX_resultant_val(GEN f, GEN g, GEN p, long M)` returns the valuation at p of $\mathrm{Res}(f, g)$. Assume f, g are both `ZX`, and that p is a prime number coprime to the leading coefficient of f. Proceeds by dynamically increasing the p-adic accuracy. To avoid an infinite loop when the resultant is 0, we return M if the Sylvester matrix mod p^M still does not have maximal rank.

`GEN ZpX_gcd(GEN f, GEN g, GEN p, GEN pm)` f a monic `ZX`, g a `ZX`, `pm` $= p^m$ a prime power. There is a unique integer $r \geq 0$ and a monic $h \in \mathbf{Q}_p[X]$ such that

$$p^r h\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X] = f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X].$$

Return the 0 polynomial if $r \geq m$ and a monic $h \in \mathbf{Z}[1/p][X]$ otherwise (whose valuation at p is $> -m$).

`GEN ZpX_reduced_resultant(GEN f, GEN g, GEN p, GEN pm)` f a monic `ZX`, g a `ZX`, `pm` $= p^m$ a prime power. The p-adic *reduced resultant* of f and g is 0 if f, g not coprime in $\mathbf{Z}_p[X]$, and otherwise the generator of the form p^d of

$$(f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo p^m.

`GEN ZpX_reduced_resultant_fast(GEN f, GEN g, GEN p, long M)` f a monic `ZX`, g a `ZX`, p a prime. Returns the p-adic reduced resultant of f and g modulo p^M. This function computes resultants for a sequence of increasing p-adic accuracies (up to M p-adic digits), returning as soon as it obtains a nonzero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

GEN ZpX_monic_factor(GEN f, GEN p, long M) f a monic ZX, p a prime, return the p-adic factorization of f, modulo p^M. This is the underlying low-level recursive function behind factorpadic (using a combination of Round 4 factorization and Hensel lifting); the factors are not sorted and the function is shallow.

GEN ZpX_primedec(GEN T, GEN p) T a monic separable ZX, p a prime, return as a factorization matrix the shape of the prime ideal decomposition of (p) in $\mathbf{Q}[X]/(T)$: the first column contains inertia degrees, the second columns contains ramification degrees.

7.3.44 ZpXQ.

GEN ZpXQ_invlift(GEN b, GEN a, GEN T, GEN p, long e) let p be a prime t_INT, a be a FpXQ (modulo (p,T)) and b a ZpXQ such that $ab \equiv 1 \bmod (p,T)$. Returns an ZpXQ A such that $A \equiv a \pmod p$ and $Ab \equiv 1 \bmod (p^e,T)$.

GEN ZpXQ_inv(GEN b, GEN T, GEN p, long e) let p be a prime t_INT and b be a FpXQ (modulo T,p^e). Returns an FpXQ A such that $Ab \equiv 1 \bmod (p^e,T)$.

GEN ZpXQ_div(GEN a, GEN b, GEN T, GEN q, GEN p, long e) let p be a prime t_INT and a and b be a FpXQ (modulo T,p^e). Returns an FpXQ c such that $cb \equiv a \bmod (p^e,T)$. The parameter q must be equal to p^e.

GEN ZpXQ_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e) let n,p be t_INTs, with $n,p > 1$ and p coprime to n, and a,b be FpXQs (modulo T) such that $a^n \equiv b \bmod (p,T)$. Returns an Fq A such that $A^n \equiv b \bmod (p^e,T)$.

GEN ZpXQ_sqrt(GEN b, GEN T, GEN p, long e) let p being a odd prime and b be a FpXQ (modulo T,p^e), returns a such that $a^2 \equiv b \bmod (p^e,T)$.

GEN ZpX_ZpXQ_liftroot(GEN f, GEN a, GEN T, GEN p, long e) as ZpXQX_liftroot, but f is a polynomial in $\mathbf{Z}[X]$.

GEN ZpX_ZpXQ_liftroot_ea(GEN f, GEN a, GEN T, GEN p, long e, void *E, GEN early(void *E, GEN x, GEN q)) as ZpX_ZpXQ_liftroot with early abort: the function early(E,x,q) will be called with x is a root of f modulo $q = p^n$ for some n. If early returns a non-NULL value z, the function returns z immediately.

GEN ZpXQ_log(GEN a, GEN T, GEN p, long e) T being a ZpX irreducible modulo p, return the logarithm of a in $\mathbf{Z}_p[X]/(T)$ to precision e, assuming that $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$ if p odd or $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$ if $p = 2$.

7.3.45 Zq.

GEN Zq_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e)

7.3.46 ZpXQM.

GEN ZpXQM_prodFrobenius(GEN M, GEN T, GEN p, long e) returns the product of matrices $M\sigma(M)\sigma^2(M)\dots\sigma^{n-1}(M)$ to precision e where σ is the lift of the Frobenius automorphism over $\mathbf{Z}_p[X]/(T)$ and n is the degree of T.

7.3.47 `ZpXQX`.

`GEN ZpXQX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)` is the routine underlying `polhensellift`. Here, p is prime, $T(Y)$ defines a finite field $\mathbf{F}_q$. A is a polynomial in $\mathbf{Z}[X,Y]$, whose leading coefficient is nonzero in $\mathbf{F}_q$. B is a vector of monic or `FqX`, pairwise coprime in $\mathbf{F}_q[X]$, whose product is congruent to $A/\mathrm{lc}(A)$ in $\mathbf{F}_q[X]$. Lifts the elements of B mod `pe` $= p^e$, such that the congruence now holds mod (T, p^e).

`GEN ZpXQX_liftroot(GEN f, GEN a, GEN T, GEN p, long e)` as `ZpX_liftroot`, but f is now a polynomial in $\mathbf{Z}[X,Y]$ and lift the root a in the unramified extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_p[Y]/(T)$, assuming $v_p(f(a)) > 0$ and $v_p(f'(a)) = 0$.

`GEN ZpXQX_liftroot_vald(GEN f, GEN a, long v, GEN T, GEN p, long e)` returns the foots of f as `ZpXQX_liftroot`, where v is the valuation of the content of f' and it is required that $v_p(f(a)) > v$ and $v_p(f'(a)) = v$.

`GEN ZpXQX_roots(GEN F, GEN T, GEN p, long e)`

`GEN ZpXQX_liftroots(GEN F, GEN S, GEN T, GEN p, long e)`

`GEN ZpXQX_divrem(GEN x, GEN Sp, GEN T, GEN q, GEN p, long e, GEN *pr)` as `FpXQX_divrem`. The parameter q must be equal to p^e.

`GEN ZpXQX_digits(GEN x, GEN B, GEN T, GEN q, GEN p, long e)` As `FpXQX_digits`. The parameter q must be equal to p^e.

`GEN ZpXQX_ZpXQXQ_liftroot(GEN f, GEN a, GEN S, GEN T, GEN p, long e)` as `ZpXQX_liftroot`, except that a is an element of $\mathbf{Z}_p[X,Y]/(S(X,Y), T(X))$.

7.3.48 `ZqX`. `ZqX` are either `ZpX` or `ZpXQX` depending whether `T` is `NULL` or not.

`GEN ZqX_roots(GEN F, GEN T, GEN p, long e)`

`GEN ZqX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)`

`GEN ZqX_liftroot(GEN f, GEN a, GEN T, GEN p, long e)`

`GEN ZqX_ZqXQ_liftroot(GEN f, GEN a, GEN P, GEN T, GEN p, long e)`

7.3.49 `ZpXQXQ`.

`GEN ZpXQXQ_invlift(GEN b, GEN a, GEN S, GEN T, GEN p, long e)` let p be a prime `t_INT`, a be a `FpXQXQ` (modulo (S,T,p)) and b a `ZpXQXQ` such that $ab \equiv 1 \bmod (S,T,p)$. Returns an `ZpXQXQ` A such that $A \equiv a \pmod{p}$ and $Ab \equiv 1 \bmod (p^e, T)$.

`GEN ZpXQXQ_inv(GEN b, GEN S, GEN T, GEN p, long e)` let p be a prime `t_INT` and b be a `ZpXQXQ` (modulo S,T,p^e). Returns a `ZpXQXQ` A such that $Ab \equiv 1 \bmod (S,T,p^e)$.

`GEN ZpXQXQ_div(GEN a, GEN b, GEN S, GEN T, GEN q, GEN p, long e)` let p be a prime `t_INT` and a and b be a `ZpXQXQ` (modulo S,T,p^e). Returns a `ZpXQXQ` c such that $cb \equiv a \bmod (S,T,p^e)$. The parameter q must be equal to p^e.

7.3.50 Other p-adic functions.

`GEN ZpM_echelon(GEN M, long early_abort, GEN p, GEN pm)` given a `ZM` M, a prime p and `pm` $= p^m$, returns an echelon form E for M mod p^m. I.e. there exist a square integral matrix U with $\det U$ coprime to p such that $E = MU$ modulo p^m. I `early_abort` is nonzero, return NULL as soon as one pivot in the echelon form is divisible by p^m. The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

`GEN zlm_echelon(GEN M, long early_abort, ulong p, ulong pm)` variant of `ZpM_echelon`, for a `Zlm` M.

`GEN ZlM_gauss(GEN a, GEN b, ulong p, long e, GEN C)` as `gauss` with the following peculiarities: a and b are `ZM`, such that a is invertible modulo p. Optional C is an `Flm` that is an inverse of $a \bmod p$ or `NULL`. Return the matrix x such that $ax = b \bmod p^e$ and all elements of x are in $[0, p^e - 1]$. For efficiency, it is better to reduce a and b mod p^e first.

`GEN padic_to_Q(GEN x)` truncate the `t_PADIC` to a `t_INT` or `t_FRAC`.

`GEN padic_to_Q_shallow(GEN x)` shallow version of `padic_to_Q`

`GEN QpV_to_QV(GEN v)` apply `padic_to_Q_shallow`

`long padicprec(GEN x, GEN p)` returns the absolute p-adic precision of the object x, by definition the minimum precision of the components of x. For a nonzero `t_PADIC`, this returns `valp(x) + precp(x)`.

`long padicprec_relative(GEN x)` returns the relative p-adic precision of the `t_INT`, `t_FRAC`, or `t_PADIC` x (minimum precision of the components of x for `t_POL` or vector/matrices). For a `t_PADIC`, this returns `precp(x)` if $x \neq 0$, and 0 for $x = 0$.

7.3.50.1 low-level.

The following technical function returns an optimal sequence of p-adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy $n \geq 1$, starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy l one iteration outputs a result correct to accuracy $2l$. For instance, to reach $n = 9$, we want to use accuracies $[1, 2, 3, 5, 9]$ instead of $[1, 2, 4, 8, 9]$. The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let $a_0 = 1$, $a_1 = 2, \ldots, a_k = n$, be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the a_i explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store a_i implicitly in a bitmask `MASK`: let $a_0 = 1$, if the i-th bit of the mask is set, set $a_{i+1} = 2a_i - 1$, and $2a_i$ otherwise; in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length $k + 1$, the function returns `MASK` $+ 2^{k+1}$, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
```

```
long l = 1;
while (mask > 1) {            /* here, the result is known to accuracy l */
  l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
  mask >>= 1;                 /* pop low order bit */
  /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the 2^{k+1} that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.

7.3.51 Conversions involving single precision objects.

7.3.51.1 To single precision.

`ulong Rg_to_Fl(GEN z, ulong p)`, `z` which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT`, a `t_INTMOD` whose modulus is divisible by p, a `t_FRAC` whose denominator is coprime to p, or a `t_PADIC` with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized, as an `Fl`.

`ulong Rg_to_F2(GEN z)`, as `Rg_to_Fl` for $p = 2$.

`ulong padic_to_Fl(GEN x, ulong p)` special case of `Rg_to_Fl`, for a x a `t_PADIC`.

`GEN RgX_to_F2x(GEN x)`, `x` a `t_POL`, returns the `F2x` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgX_to_Flx(GEN x, ulong p)`, `x` a `t_POL`, returns the `Flx` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgXV_to_FlxV(GEN x, ulong p)`, `x` a vector, returns the `FlxV` obtained by applying `RgX_to_Flx` coefficientwise.

`GEN Rg_to_F2xq(GEN z, GEN T)`, `z` a `GEN` which can be mapped to $\mathbf{F}_2[X]/(T)$: anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_F2x` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `F2x`), a suitable `t_RFRAC`. Returns `z` as an `F2xq`, normalized.

`GEN Rg_to_Flxq(GEN z, GEN T, ulong p)`, `z` a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns `z` as an `Flxq`, normalized.

`GEN RgX_to_FlxqX(GEN z, GEN T, ulong p)`, `z` a `GEN` which can be mapped to $\mathbf{F}_p[x]/(T)[X]$: anything `Rg_to_Flxq` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns `z` as an `FlxqX`, normalized.

`GEN ZX_to_Flx(GEN x, ulong p)` reduce `ZX` `x` modulo `p` (yielding an `Flx`). Faster than `RgX_to_Flx`.

`GEN ZV_to_Flv(GEN x, ulong p)` reduce `ZV x` modulo `p` (yielding an `Flv`).

`GEN ZXV_to_FlxV(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the vector's coefficients.

`GEN ZXT_to_FlxT(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the tree leaves.

`GEN ZXX_to_FlxX(GEN B, ulong p, long v)`, as `ZX_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN zxX_to_FlxX(GEN z, ulong p)` as `zx_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN ZXXV_to_FlxXV(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the vector's coefficients.

`GEN ZXXT_to_FlxXT(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the tree leaves.

`GEN RgV_to_Flv(GEN x, ulong p)` reduce the `t_VEC`/`t_COL` x modulo p, yielding a `t_VECSMALL`.

`GEN RgM_to_Flm(GEN x, ulong p)` reduce the `t_MAT` x modulo p.

`GEN ZM_to_Flm(GEN x, ulong p)` reduce `ZM` x modulo p (yielding an `Flm`).

`GEN ZVV_to_FlvV(GEN x, ulong p)` reduce `ZVV` x modulo p (yielding an `FlvV`).

`GEN ZV_to_zv(GEN z)`, converts coefficients using `itos`

`GEN ZV_to_nv(GEN z)`, converts coefficients using `itou`

`GEN ZM_to_zm(GEN z)`, converts coefficients using `itos`

7.3.51.2 From single precision.

`GEN Flx_to_ZX(GEN z)`, converts to `ZX` (`t_POL` of nonnegative `t_INT`s in this case)

`GEN Flx_to_FlxX(GEN z)`, converts to `FlxX` (`t_POL` of constant `Flx` in this case).

`GEN Flx_to_ZX_inplace(GEN z)`, same as `Flx_to_ZX`, in place (`z` is destroyed).

`GEN FlxX_to_ZXX(GEN B)`, converts an `FlxX` to a polynomial with `ZX` or `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN FlxXC_to_ZXXC(GEN B)`, converts an `FlxXC` to a `t_COL` with `ZXX` coefficients (repeated calls to `FlxX_to_ZXX`).

`GEN FlxXM_to_ZXXM(GEN B)`, converts an `FlxXM` to a `t_MAT` with `ZXX` coefficients (repeated calls to `FlxX_to_ZXX`).

`GEN FlxC_to_ZXC(GEN x)`, converts a vector of `Flx` to a column vector of polynomials with `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN FlxV_to_ZXV(GEN x)`, as above but return a `t_VEC`.

`void F2xV_to_FlxV_inplace(GEN v)` v is destroyed.

`void F2xV_to_ZXV_inplace(GEN v)` v is destroyed.

`void FlxV_to_ZXV_inplace(GEN v)` v is destroyed.

`GEN FlxM_to_ZXM(GEN z)`, converts a matrix of `Flx` to a matrix of polynomials with `t_INT` coefficients (repeated calls to `Flx_to_ZX`).

`GEN zx_to_ZX(GEN z)`, as `Flx_to_ZX`, without assuming the coefficients to be nonnegative.

`GEN zx_to_Flx(GEN z, ulong p)` as `Flx_red` without assuming the coefficients to be nonnegative.

`GEN Flc_to_ZC(GEN z)`, converts to `ZC` (`t_COL` of nonnegative `t_INT`s in this case)

`GEN Flc_to_ZC_inplace(GEN z)`, same as `Flc_to_ZC`, in place (`z` is destroyed).

`GEN Flv_to_ZV(GEN z)`, converts to `ZV` (`t_VEC` of nonnegative `t_INT`s in this case)

`GEN Flm_to_ZM(GEN z)`, converts to `ZM` (`t_MAT` with nonnegative `t_INT`s coefficients in this case)

`GEN Flm_to_ZM_inplace(GEN z)`, same as `Flm_to_ZM`, in place (`z` is destroyed).

`GEN zc_to_ZC(GEN z)` as `Flc_to_ZC`, without assuming coefficients are nonnegative.

`GEN zv_to_ZV(GEN z)` as `Flv_to_ZV`, without assuming coefficients are nonnegative.

`GEN zm_to_ZM(GEN z)` as `Flm_to_ZM`, without assuming coefficients are nonnegative.

`GEN zv_to_Flv(GEN z, ulong p)`

`GEN zm_to_Flm(GEN z, ulong p)`

7.3.51.3 Mixed precision linear algebra. Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

`GEN RgM_zc_mul(GEN x, GEN y)`

`GEN RgMrow_zc_mul(GEN x, GEN y, long i)`

`GEN RgM_zm_mul(GEN x, GEN y)`

`GEN RgV_zc_mul(GEN x, GEN y)`

`GEN RgV_zm_mul(GEN x, GEN y)`

`GEN ZM_zc_mul(GEN x, GEN y)`

`GEN zv_ZM_mul(GEN x, GEN y)`

`GEN ZV_zc_mul(GEN x, GEN y)`

`GEN ZM_zm_mul(GEN x, GEN y)`

`GEN ZC_z_mul(GEN x, long y)`

`GEN ZM_nm_mul(GEN x, GEN y)` the entries of y are `ulong`s.

`GEN nm_Z_mul(GEN y, GEN c)` the entries of y are `ulong`s.

7.3.51.4 Miscellaneous involving Fl.

`GEN Fl_to_Flx(ulong x, long evx)` converts a `unsigned long` to a scalar `Flx`. Assume that `evx = evalvarn(vx)` for some variable number `vx`.

`GEN Z_to_Flx(GEN x, ulong p, long sv)` converts a `t_INT` to a scalar `Flx` polynomial. Assume that `sv = evalvarn(v)` for some variable number `v`.

`GEN Flx_to_Flv(GEN x, long n)` converts from `Flx` to `Flv` with `n` components (assumed larger than the number of coefficients of `x`).

`GEN zx_to_zv(GEN x, long n)` as `Flx_to_Flv`.

`GEN Flv_to_Flx(GEN x, long sv)` converts from vector (coefficient array) to (normalized) polynomial in variable v.

`GEN zv_to_zx(GEN x, long n)` as `Flv_to_Flx`.

`GEN Flm_to_FlxV(GEN x, long sv)` converts the columns of `Flm x` to an array of `Flx` in the variable v (repeated calls to `Flv_to_Flx`).

`GEN FlxM_to_FlxXV(GEN V, long v)` see `RgM_to_RgXV`

`GEN zm_to_zxV(GEN x, long n)` as `Flm_to_FlxV`.

`GEN Flm_to_FlxX(GEN x, long sw, long sv)` same as `Flm_to_FlxV(x,sv)` but returns the result as a (normalized) polynomial in variable w.

`GEN FlxV_to_Flm(GEN v, long n)` reverse `Flm_to_FlxV`, to obtain an `Flm` with `n` rows (repeated calls to `Flx_to_Flv`).

`GEN FlxX_to_Flx(GEN P)` Let $P(x, X)$ be a `FlxX`, return $P(0, X)$ as a `Flx`.

`GEN FlxX_to_Flm(GEN v, long n)` reverse `Flm_to_FlxX`, to obtain an `Flm` with `n` rows (repeated calls to `Flx_to_Flv`).

`GEN FlxX_to_FlxC(GEN B, long n, long sv)` see `RgX_to_RgV`. The coefficients of `B` are assumed to be in the variable v.

`GEN FlxV_to_FlxX(GEN x, long v)` see `RgV_to_RgX`.

`GEN FlxXV_to_FlxM(GEN V, long n, long sv)` see `RgXV_to_RgM`. The coefficients of `V[i]` are assumed to be in the variable v.

`GEN Fly_to_FlxY(GEN a, long sv)` convert coefficients of `a` to constant `Flx` in variable v.

7.3.51.5 Miscellaneous involving F2x.

`GEN F2x_to_F2v(GEN x, long n)` converts from `F2x` to `F2v` with `n` components (assumed larger than the number of coefficients of `x`).

`GEN F2xC_to_ZXC(GEN x)`, converts a vector of `F2x` to a column vector of polynomials with `t_INT` coefficients (repeated calls to `F2x_to_ZX`).

`GEN F2xC_to_FlxC(GEN x)`

`GEN FlxC_to_F2xC(GEN x)`

`GEN F2xV_to_F2m(GEN v, long n)` `F2x_to_F2v` to each polynomial to get an `F2m` with `n` rows.

7.4 Higher arithmetic over Z: primes, factorization.

7.4.1 Pure powers.

`long Z_issquare(GEN n)` returns 1 if the `t_INT` n is a square, and 0 otherwise. This is tested first modulo small prime powers, then `sqrtremi` is called.

`long Z_issquareall(GEN n, GEN *sqrtn)` as `Z_issquare`. If n is indeed a square, set `sqrtn` to its integer square root. Uses a fast congruence test mod $64 \times 63 \times 65 \times 11$ before computing an integer square root.

`long Z_ispow2(GEN x)` returns 1 if the `t_INT` x is a power of 2, and 0 otherwise.

`long uissquare(ulong n)` as `Z_issquare`, for an `ulong` operand `n`.

`long uissquareall(ulong n, ulong *sqrtn)` as `Z_issquareall`, for an `ulong` operand `n`.

`ulong usqrt(ulong a)` returns the floor of the square root of a.

`ulong usqrtn(ulong a, ulong n)` returns the floor of the n-th root of a.

`long Z_ispower(GEN x, ulong k)` returns 1 if the `t_INT` n is a k-th power, and 0 otherwise; assume that $k > 1$.

`long Z_ispowerall(GEN x, ulong k, GEN *pt)` as `Z_ispower`. If n is indeed a k-th power, set `*pt` to its integer k-th root.

`long Z_isanypower(GEN x, GEN *ptn)` returns the maximal $k \geq 2$ such that the `t_INT` $x = n^k$ is a perfect power, or 0 if no such k exist; in particular `ispower(1)`, `ispower(0)`, `ispower(-1)` all return 0. If the return value k is not 0 (so that $x = n^k$) and `ptn` is not `NULL`, set `*ptn` to n.

The following low-level functions are called by `Z_isanypower` but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer $x > 0$ is a 3-rd, 5-th or 7-th power. The bits of `*mask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. $*$`pmask` $= 7$ performs all tests). They are updated during the call: if the "i-th power" bit is set to 0 then x is not a k-th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an i-th power bit is initially set to 0, we take it at face value and assume x is not an i-th power without performing any test. If the return value k is nonzero, set `*ptn` to n such that $x = n^k$.

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let $x > 0$ be an integer, `cutoff` > 0 and T be an iterator over primes ≥ 11, we look for the smallest prime p such that $x = n^p$ (advancing T as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for $p < 11$.

Fail and return 0 when the existence of p would imply $2^{\texttt{cutoff}} > x^{1/p}$, meaning that a possible n is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power p^i and set `*ptn` to the p^i-th root of x (which is now not a p-th power). We may immediately recall the function with the same parameters after setting $x = *$`ptn`: it will start at the next prime.

7.4.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT` `n`. The "primes" in the factorization are actually strong pseudoprimes.

`GEN absZ_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT` n are less or equal to lim.

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns `NULL` if a prime factor of the `t_INT` n is $> lim$, and returns the factorization of n otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Shallow function.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than `lim`. The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide n by all primes $p <$ `lim` in the precomputed list of prime numbers and the `addprimes` prime table. Return the corresponding factorization matrix. The first column of the factorization matrix may contain a single composite, which may or may not be the last entry in presence of a prime table.

If `lim` $= 0$, the effect is the same as setting `lim` $=$ `maxprime`$() + 1$: use all precomputed primes.

`int Z_has_prime_3mod4(GEN n)` return 1 if and only if the non-zero integer n has a prime divisor congruent to 3 modulo 4.

`GEN absZ_factor_limit(GEN n, ulong all)` returns `Z_factor_limit(absi(n))`.

`GEN absZ_factor_limit_strict(GEN n, ulong lim, GEN *pU)`. This function is analogous to `absZ_factor_limit`, with a better interface: trial divide n by all primes $p <$ `lim` (supposedly in the precomputed list of prime numbers, but `lim` may be larger) and then the full `addprimes` prime table (whatever their size). Return the corresponding factorization matrix. In this case, a composite cofactor is *not* included.

If `pU` is not `NULL`, set it to the cofactor, which is either `NULL` (no cofactor) or $[q, k]$, where $q >$ `lim` and $k > 0$. The prime divisors of q are greater than `lim` and not in the `addprimes` table, q is not a pure power, and q^k is the largest power of q dividing n. It may happen that q is prime.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoothen(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VEC` L containing a list of primes and a `t_INT` n, trial divide n by the elements of L and return the cofactor. Return `NULL` if the cofactor is ± 1. `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALL`s. Shallow function.

`GEN Z_lsmoothen(GEN n, GEN L, GEN *pP, GEN *pE)` as `Z_smoothen` where L is a `t_VECSMALL` of small primes and both `*P` and `*E` are given as `t_VECSMALL`.

`GEN Z_factor_listP(GEN N, GEN L)` given a `t_INT` N, a vector or primes L containing all prime divisors of N (and possibly others). Return `factor(N)`. Shallow function.

`GEN factor_pn_1(GEN p, ulong n)` returns the factorization of $p^n - 1$, where p is prime and n is a positive integer.

`GEN factor_pn_1_limit(GEN p, ulong n, ulong B)` returns a partial factorization of $p^n - 1$, where p is prime and n is a positive integer. Don't actively search for prime divisors $p > B$, but

we may find still find some due to Aurifeuillian factorizations. Any entry $> B^2$ in the output factorization matrix is *a priori* not a prime (but may well be).

`GEN factor_Aurifeuille_prime(GEN p, long n)` an Aurifeuillian factor of $\phi_n(p)$, assuming p prime and an Aurifeuillian factor exists ($p\zeta_n$ is a square in $\mathbf{Q}(\zeta_n)$).

`GEN factor_Aurifeuille(GEN a, long n)` an Aurifeuillian factor of $\phi_n(a)$, assuming a is a nonzero integer and $n > 2$. Returns 1 if no Aurifeuillian factor exists.

`GEN odd_prime_divisors(GEN a)` `t_VEC` of all prime divisors of the `t_INT` a.

`GEN factoru(ulong n)`, returns the factorization of n. The result is a 2-component vector $[P, E]$, where P and E are `t_VECSMALL` containing the prime divisors of n, and the $v_p(n)$.

`GEN factoru_pow(ulong n)`, returns the factorization of n. The result is a 3-component vector $[P, E, C]$, where P, E and C are `t_VECSMALL` containing the prime divisors of n, the $v_p(n)$ and the $p^{v_p(n)}$.

`GEN vecfactoru(ulong a, ulong b)`, returns a `t_VEC` v containing the factorizations (`factoru` format) of $a, \dots, b$; assume that $b \geq a > 0$. Uses a sieve with primes up to $\sqrt{b}$. For all c, $a \leq c \leq b$, the factorization of c is given in $v[c - a + 1]$.

`GEN vecfactoroddu(ulong a, ulong b)`, returns a `t_VEC` v containing the factorizations (`factoru` format) of odd integers in $a, \dots, b$; assume that $b \geq a > 0$ are odd. Uses a sieve with primes up to $\sqrt{b}$. For all odd c, $a \leq c \leq b$, the factorization of c is given in in $v[(c - a)/2 + 1]$.

`GEN vecfactoru_i(ulong a, ulong b)`, private version of `vecfactoru`, not stack-clean.

`GEN vecfactoroddu_i(ulong a, ulong b)`, private version of `vecfactoroddu`, not stack-clean.

`GEN vecfactorsquarefreeu(ulong a, ulong b)` return a `t_VEC` v containing the prime divisors of squarefree integers in $a, \dots, b$; assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$. For all squarefree c, $a \leq c \leq b$, the prime divisors of c (as a `t_VECSMALL`) are given in $v[c - a + 1]$, and the other entries are `NULL`. Note that because of these `NULL` markers, v is not a valid `GEN`, it is not stack-clean.

`GEN vecfactorsquarefreeu_coprime(ulong a, ulong b, GEN P)` given a *sorted* `t_VECSMALL` of primes P, return a `t_VEC` v containing the prime divisors of squarefree integers in $a, \dots, b$ coprime to the elements of P; assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$. For all squarefree c, $a \leq c \leq b$, the prime divisors of c (as a `t_VECSMALL`) are given in $v[c - a + 1]$, and the other entries are `NULL`. Note that because of these `NULL` markers, v is not a valid `GEN`, it is not stack-clean.

`GEN vecsquarefreeu(ulong a, ulong b)` return a `t_VECSMALL` v containing the squarefree integers in $a, \dots, b$. Assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$.

`ulong tridiv_bound(GEN n)` returns the trial division bound used by `Z_factor`(n).

`GEN tridiv_boundu(ulong n)` returns the trial division bound used by `factoru`n.

`GEN Z_pollardbrent(GEN N, long n, long seed)` try to factor `t_INT` N using $n \geq 1$ rounds of Pollard iterations; *seed* is an integer whose value (mod 8) selects the quadratic polynomial use to generate Pollard's (pseudo)random walk. Returns `NULL` on failure, else a vector of 2 (possibly 3) integers whose product is N.

`GEN Z_ECM(GEN N, long n, long seed, ulong B1)` try to factor `t_INT` N using $n \geq 1$ rounds of ECM iterations (on 8 to 64 curves simultaneously, depending on the size of N); *seed* is an integer whose value selects the curves to be used: increase it by $64n$ to make sure that a subsequent call

with a factor of N uses a disjoint set of curves. Finally $B_1 > 7$ determines the computations performed on the curves: we compute $[k]P$ for some point in $E(\mathbf{Z}/N\mathbf{Z})$ and $k = q\prod p^{e_p}$ where $p^{e_p} \leq B_1$ and $q \leq B_2 := 110B_1$; a higher value of B_1 means higher chances of hitting a factor and more time spent. The computation is deterministic for a given set of parameters. Returns `NULL` on failure, else a nontrivial factor or `N`.

`GEN Q_factor(GEN x)` as `Z_factor`, where x is a `t_INT` or a `t_FRAC`.

`GEN Q_factor_limit(GEN x, ulong lim)` as `Z_factor_limit`, where x is a `t_INT` or a `t_FRAC`.

7.4.3 Coprime factorization.

Given a and b two nonzero integers, let $\mathbf{ppi}(a,b)$, $\mathbf{ppo}(a,b)$, $\mathbf{ppg}(a,b)$, $\mathbf{pple}(a,b)$ (powers in a of primes inside b, outside b, greater than those in b, less than or equal to those in b) be the integers defined by

- $v_p(\text{ppi}) = v_p(a)[v_p(b) > 0]$,
- $v_p(\text{ppo}) = v_p(a)[v_p(b) = 0]$,
- $v_p(\text{ppg}) = v_p(a)[v_p(a) > v_p(b)]$,
- $v_p(\text{pple}) = v_p(a)[v_p(a) \leq v_p(b)]$.

`GEN Z_ppo(GEN a, GEN b)` returns $\text{ppo}(a,b)$; shallow function.

`ulong u_ppo(ulong a, ulong b)` returns $\text{ppo}(a,b)$.

`GEN Z_ppgle(GEN a, GEN b)` returns $[\text{ppg}(a,b), \text{pple}(a,b)]$; shallow function.

`GEN Z_ppio(GEN a, GEN b)` returns $[\gcd(a,b), \text{ppi}(a,b), \text{ppo}(a,b)]$; shallow function.

`GEN Z_cba(GEN a, GEN b)` fast natural coprime base algorithm. Returns a vector of coprime divisors of a and b such that both a and b can be multiplicatively generated from this set. Perfect powers are not removed, use `Z_isanypower` if needed; shallow function.

`GEN ZV_cba_extend(GEN P, GEN b)` extend a coprime basis P by the integer b, the result being a coprime basis for $P \cup \{b\}$. Perfect powers are not removed; shallow function.

`GEN ZV_cba(GEN v)` given a vector of nonzero integers v, return a coprime basis for v. Perfect powers are not removed; shallow function.

7.4.4 Checks attached to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer N (`t_INT`), the factorization fa of a positive integer (a `t_MAT` with two columns containing respectively primes and exponents), or a vector $[N, fa]$. A few functions accept nonzero integers (e.g. `omega`), and some others arbitrary integers (e.g. `factorint`, ...).

`int is_Z_factorpos(GEN f)` returns 1 if f looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column ("primes") are prime, or even pairwise coprime, nor that they are stricly increasing.

`int is_Z_factornon0(GEN f)` returns 1 if f looks like the factorization of a nonzero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to `is_Z_factorpos`. (Entries in the first column need only be nonzero integers.)

`int is_Z_factor(GEN f)` returns 1 if f looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are integers. Entries in the second column ("exponents") are all positive. Either it encodes the "factorization" 0^e, $e > 0$, or entries in the first column ("primes") are all nonzero.

`GEN clean_Z_factor(GEN f)` assuming f is the factorization of an integer n, return the factorization of $|n|$, i.e. remove -1 from the factorization. Shallow function.

`GEN fuse_Z_factor(GEN f, GEN B)` assuming f is the factorization of an integer n, return `boundfact(n, B)`, i.e. return a factorization where all primary factors for $|p| \leq B$ are preserved, and all others are "fused" into a single composite integer; if that remainder is trivial, i.e. equal to 1, it is of course not included. Shallow function.

In the following three routines, f is the name of an arithmetic function, and n a supplied argument. They all raise exceptions if n does not correspond to an integer or an integer factorization of the expected shape.

`GEN check_arith_pos(GEN n, const char *f)` check whether n is attached to the factorization of a positive integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_DOMAIN` ($n \leq 0$) or an `e_TYPE` exception (other failures).

`GEN check_arith_non0(GEN n, const char *f)` check whether n is attached to the factorization of a nonzero integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_TYPE` exception.

`GEN check_arith_all(GEN n, const char *f)` is attached to the factorization of an integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise.

7.4.5 Incremental integer factorization.

Routines attached to the dynamic factorization of an integer n, iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius`(n) can be trivially computed if we hit p such that $p^2 \mid n$. For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime $< 2^{14}$ divides n.

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer n. If `moebius` is nonzero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that $n > 1$ is a *composite* `t_INT` whose prime divisors satisfy $p > 2^{14}$ *and* that one can write to n in place.

This function stores data on the stack, no garbage collection call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor p^e from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to p.

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case p is set to `NULL`. If we ran in Moebius mode and the factorization was in fact aborted, we have $e = 1$, otherwise $e = 0$.

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list k^e, where k is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given n whose prime divisors are $> 2^{14}$, returns the decision the factoring engine would take about the compositeness of n: 0 if n is a proven composite, and 1 if we believe it to be prime; more precisely, n is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

7.4.6 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT` `n` is square-free, and 0 otherwise.

`long Z_issquarefree_fact(GEN fa)` same, where `fa` is `factor(n)`.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT` `x` is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer d dividing n such that n/d is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return $[d, f]$ with d squarefree and $n = df^2$.

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n,lim)` to partially factor `n`. The result is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n,lim)` to partially factor `n`. The resulting d is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

7.4.7 Primes, primality and compositeness tests.

7.4.7.1 Chebyshev's π function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes $p \leq n$ (Chebyshev's π function).

`double primepi_upper_bound(double x)` return a quick upper bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

7.4.7.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime $\geq n$. Return 0 if it cannot be represented as an `ulong` (n bigger than $2^{64} - 59$ or $2^{32} - 5$ depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime $\leq n$. Return 0 if $n \leq 1$.

`ulong uprime(long n)` returns the n-th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first m primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first m primes, as a `t_VEC` of `t_INT`s.

`GEN primes_interval(GEN a, GEN b)` return the primes in the interval $[a, b]$, as a `t_VEC` of `t_INT`s.

`GEN primes_interval_zv(ulong a, ulong b)` return the primes in the interval $[a, b]$, as a `t_VECSMALL` of `ulong`s.

`GEN primes_upto_zv(ulong b)` return the primes in the interval $[2, b]$, as a `t_VECSMALL` of `ulong`s.

`GEN mpprimorial(long n)` return $n\#$, the product of primes less than or equal to n. The integer n must be non-negative.

7.4.7.3 Tests.

`int uisprime(ulong p)`, returns 1 if `p` is a prime number and 0 otherwise.

`int uisprime_101(ulong p)`, assuming that p has no divisor ≤ 101, returns 1 if `p` is a prime number and 0 otherwise.

`int uisprime_661(ulong p)`, assuming that p has no divisor ≤ 661, returns 1 if `p` is a prime number and 0 otherwise.

`int isprime(GEN n)`, returns 1 if the `t_INT` `n` is a (fully proven) prime number and 0 otherwise.

`long isprimeAPRCL(GEN n)`, returns 1 if the `t_INT` `n` is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if nonprimes are included!

`long isprimeECPP(GEN n)`, returns 1 if the `t_INT` `n` is a prime number and 0 otherwise, using only the ECPP test. The workhorse `isprime` should be faster on average.

`long BPSW_psp(GEN n)`, returns 1 if the `t_INT` `n` is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

`int BPSW_isprime(GEN x)` assuming x is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

```
BPSW_psp(x) && BPSW_isprime(x)
```

`long millerrabin(GEN n, long k)` performs k strong Rabin-Miller compositeness tests on the `t_INT` n, using k random bases. This function also caches square roots of -1 that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored n at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for $k \geq 4$ and we expect it to be less reliable.

`GEN ecpp(GEN N)` returns an ECPP certificate for `t_INT` N; underlies `primecert`.

`GEN ecpp0(GEN N, long t)` returns a (potentially) partial ECPP certificate for `t_INT` N where strong pseudo-primes $< 2^t$ are included as primes in the certificate. Underlies `primecert` with t set to the `partial` argument.

`GEN ecppexport(GEN cert, long flag)` export a PARI ECPP certificate to MAGMA or Primo format; underlies `primecertexport`.

`long ecppisvalid(GEN cert)` checks whether a PARI ECPP certificate is valid; underlies `primecertisvalid`.

`long check_ecppcert(GEN cert)` checks whether `cert` looks like a PARI ECPP certificate, (valid or invalid) without doing any computation.

7.4.8 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator T over primes in $[a, b]$; over primes $\geq a$ if $b =$ `NULL`. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator T over primes in an arithmetic progression in $[a, b]$; over primes $\geq a$ if $b =$ `NULL`. The argument q is either a `t_INT` ($p \equiv a \pmod q$) or a `t_INTMOD` `Mod(c,N)` and we restrict to that congruence class. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that T was initialized by `forprime_init`.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)`

`ulong u_forprime_next(forprime_t *T)`

`void u_forprime_restrict(forprime_t *T, ulong c)` let T an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and $a \leq c \leq b$. Restrict the range of primes considered to $[a, c]$.

`int u_forprime_arith_init(forprime_t *T, ulong a, ulong b, ulong c, ulong q)` initialize an iterator over primes in $[a, b]$, congruent to c modulo q. Subsequent calls to `u_forprime_next` will only return primes congruent to c modulo q. Note that unless $(c, q) = 1$ there will be at most one such prime.

7.5 Integral, rational and generic linear algebra.

7.5.1 `ZC / ZV, ZM`. A `ZV` (resp. a `ZM`, resp. a `ZX`) is a `t_VEC` or `t_COL` (resp. `t_MAT`, resp. `t_POL`) with `t_INT` coefficients.

7.5.1.1 `ZC / ZV`.

`void RgV_check_ZV(GEN x, const char *s)` Assuming `x` is a `t_VEC` or `t_COL` raise an error if it is not a `ZV` (s should point to the name of the caller).

`int RgV_is_ZV(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a `ZV`, and 0 otherwise.

`int RgV_is_ZVpos(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a `ZV` with positive entries, and 0 otherwise.

`int RgV_is_ZVnon0(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a `ZV` with nonzero entries, and 0 otherwise.

`int RgV_is_QV(GEN P)` return 1 if the `RgV` P has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int RgV_is_arithprog(GEN v, GEN *a, GEN *b)` assuming x is a `t_VEC` or `t_COL` return 1 if its entries follow an arithmetic progression of the form $a + b * n$, $n = 0, 1, \ldots$ and set a and b. Else return 0.

`int ZV_equal0(GEN x)` returns 1 if all entries of the `ZV` x are zero, and 0 otherwise.

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int ZV_abscmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order, comparing absolute values).

`int ZV_equal(GEN x, GEN y)` returns 1 if the two `ZV` are equal and 0 otherwise. A `t_COL` and a `t_VEC` with the same entries are declared equal.

`GEN identity_ZV(long n)` return the `t_VEC` $[1, 2, \ldots, n]$.

`GEN ZC_add(GEN x, GEN y)` adds `x` and `y`.

`GEN ZC_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN ZC_Z_add(GEN x, GEN y)` adds `y` to `x[1]`.

`GEN ZC_Z_sub(GEN x, GEN y)` subtracts `y` to `x[1]`.

`GEN Z_ZC_sub(GEN a, GEN x)` returns the vector $[a - x_1, -x_2, \ldots, -x_n]$.

`GEN ZC_copy(GEN x)` returns a (`t_COL`) copy of `x`.

`GEN ZC_neg(GEN x)` returns $-$`x` as a `t_COL`.

`void ZV_neg_inplace(GEN x)` negates the `ZV x` in place, by replacing each component by its opposite (the type of `x` remains the same, `t_COL` or `t_COL`). If you want to save even more memory by avoiding the implicit component copies, use `ZV_togglesign`.

`void ZV_togglesign(GEN x)` negates `x` in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the `ZC` or `ZV x` (which can be a column or row vector) by the `t_INT y`, returning a `ZC`.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZC_divexactu(GEN x, ulong y)` returns x/y assuming all divisions are exact.

`GEN ZC_Z_div(GEN x, GEN y)` returns x/y, where the resulting vector has rational entries.

`GEN ZV_ZV_mod(GEN a, GEN b)`. Assuming a and b are two `ZV` of the same length, returns the vector whose i-th component is `modii`$(a[i], b[i])$.

`GEN ZV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming x and y have `t_INT` entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming x has `t_INT` entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$, where u, v are `t_INT` and x, y are `ZC` or `ZV`. Return a `ZC`

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets $X \leftarrow X + vY$, where v is a `t_INT` and X, Y are `ZC` or `ZV`. (The result has the type of X.) Memory efficient (e.g. no-op if $v = 0$), but shallow.

`void ZC_lincomb1_inplace_i(GEN X, GEN Y, GEN v, long n)` variant of `ZC_lincomb1_inplace`: only update $X[1], \ldots, X[n]$, assuming that $n <$ `lg`(X).

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC x` (seen as a column vector) by the `ZV y` (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of `x`.

`GEN ZV_extgcd(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the content of A and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [D, 0, \ldots, 0]$.

`GEN ZV_prod(GEN x)` returns the product of all the components of `x` (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of `x` (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in x.

`long ZV_max_expi(GEN x)` returns the number of bits of the largest entry in x.

`int ZV_dvd(GEN x, GEN y)` assuming x, y are two `ZV`s of the same length, return 1 if $y[i]$ divides $x[i]$ for all i and 0 otherwise. Error if one of the $y[i]$ is 0.

`GEN ZV_sort(GEN L)` sort the `ZV` L. Returns a vector with the same type as L.

`GEN ZV_sort_shallow(GEN L)` shallow version of `ZV_sort`.

`void ZV_sort_inplace(GEN L)` sort the `ZV` L, in place.

`GEN ZV_sort_uniq(GEN L)` sort the `ZV` L, removing duplicate entries. Returns a vector with the same type as L.

`GEN ZV_sort_uniq_shallow(GEN L)` shallow version of `ZV_sort_uniq`.

`long ZV_search(GEN L, GEN y)` look for the `t_INT` y in the sorted `ZV` L. Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the `ZV` L, would sort the vector. The result is a `t_VECSMALL`.

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of x and y. Shallow function. In case two entries are equal in x and y, include the one from x.

`GEN ZC_union_shallow(GEN x, GEN y)` as `ZV_union_shallow` but return a `t_COL`.

7.5.1.2 ZM.

`void RgM_check_ZM(GEN A, const char *s)` Assuming `x` is a `t_MAT` raise an error if it is not a `ZM` (s should point to the name of the caller).

`GEN RgM_rescale_to_int(GEN x)` given a matrix x with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZM` wich is very close to Dx for some well-chosen integer D. More precisely, if the input is exact, D is the denominator of x; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN ZM_copy(GEN x)` returns a copy of `x`.

`int ZM_equal(GEN A, GEN B)` returns 1 if the two `ZM` are equal and 0 otherwise.

`int ZM_equal0(GEN A)` returns 1 if the `ZM` A is identically equal to 0.

`GEN ZM_add(GEN x, GEN y)` returns `x` + `y` (assumed to have compatible dimensions).

`GEN ZM_sub(GEN x, GEN y)` returns `x` − `y` (assumed to have compatible dimensions).

`GEN ZM_neg(GEN x)` returns −`x`.

`void ZM_togglesign(GEN x)` negates `x` in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZM_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN ZM2_mul(GEN x, GEN y)` multiplies the two-by-two `ZM` x and y.

`GEN ZM_sqr(GEN x)` returns x^2, where x is a square `ZM`.

`GEN ZM2_sqr(GEN x)` returns x^2, where x is a two-by-two `ZM`.

`GEN ZM_Z_mul(GEN x, GEN y)` multiplies the `ZM x` by the `t_INT y`.

`GEN ZM_ZC_mul(GEN x, GEN y)` multiplies the `ZM x` by the `ZC y` (seen as a column vector, assumed to have compatible dimensions).

`GEN ZM_ZX_mul(GEN x, GEN T)` returns $x \times y$, where y is `RgX_to_RgC`$(T, \texttt{lg}(x) - 1)$.

`GEN ZM_diag_mul(GEN d, GEN m)` given a vector d with integer entries and a `ZM` m of compatible dimensions, return `diagonal(d) * m`.

`GEN ZM_mul_diag(GEN m, GEN d)` given a vector d with integer entries and a `ZM` m of compatible dimensions, return `m * diagonal(d)`.

`GEN ZM_multosym(GEN x, GEN y)`

`GEN ZM_transmultosym(GEN x, GEN y)`

`GEN ZM_transmul(GEN x, GEN y)`

`GEN ZMrow_ZC_mul(GEN x, GEN y, long i)` multiplies the i-th row of `ZM x` by the `ZC y` (seen as a column vector, assumed to have compatible dimensions). Assumes that x is nonempty and $0 < i <$ `lg(x[1])`.

`int ZMrow_equal0(GEN V, long i)` returns 1 if the i-th row of the `ZM V` is zero, and 0 otherwise.

`GEN ZV_ZM_mul(GEN x, GEN y)` multiplies the `ZV x` by the `ZM y`. Returns a `t_VEC`.

`GEN ZM_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZM_divexactu(GEN x, ulong y)` returns x/y assuming all divisions are exact.

`GEN ZM_Z_div(GEN x, GEN y)` returns x/y, where the resulting matrix has rational entries.

`GEN ZM_ZV_mod(GEN a, GEN b)`. Assuming a is a `ZM` whose columns have the same length as the `ZV` b, apply `ZV_ZV_mod`$(a[i], b)$ to all columns.

`GEN ZC_Q_mul(GEN x, GEN y)` returns $x * y$, where y is a rational number and the resulting `t_COL` has rational entries.

`GEN ZM_Q_mul(GEN x, GEN y)` returns $x * y$, where y is a rational number and the resulting matrix has rational entries.

`GEN ZM_pow(GEN x, GEN n)` returns `x`$^{\texttt{n}}$, assuming `x` is a square `ZM` and `n` ≥ 0.

`GEN ZM_powu(GEN x, ulong n)` returns `x`$^{\texttt{n}}$, assuming `x` is a square `ZM` and `n` ≥ 0.

`GEN ZMV_prod(GEN v)` returns the product of the elements of v.

`GEN ZM_det(GEN M)` if `M` is a `ZM`, returns the determinant of M. This is the function underlying `matdet` whenever M is a `ZM`.

`GEN ZM_permanent(GEN M)` if `M` is a `ZM`, returns its permanent. This is the function underlying `matpermanent` whenever M is a `ZM`. It assumes that the matrix is square of dimension $<$ `BITS_IN_LONG`.

`GEN ZM_detmult(GEN M)` if `M` is a `ZM`, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying `detint`.

`GEN ZM_supnorm(GEN x)` return the sup norm of the `ZM` x.

`GEN ZM_charpoly(GEN M)` returns the characteristic polynomial (in variable 0) of the `ZM` M.

`GEN ZM_trace(GEN M)` returns the trace of the `ZM` M.

`GEN ZM_imagecompl(GEN x)` returns `matimagecompl(x)`.

`long ZM_rank(GEN x)` returns `matrank(x)`.

`GEN ZM_ker(GEN x)` returns the primitive part of `matker(x)`; in other words the $\mathbf{Q}$-basis vectors are made integral and primitive.

`GEN ZM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN ZM_indeximage(GEN x)` returns `gel(ZM_indexrank(x), 2)`.

`long ZM_max_lg(GEN x)` returns the effective length of the longest entry in x.

`long ZM_max_expi(GEN x)` returns the number of bits of the largest entry in x.

`GEN ZM_inv(GEN M, GEN *pd)` if `M` is a `ZM`, return a primitive matrix H such that MH is d times the identity and set `*pd` to d. Uses a multimodular algorithm up to Hadamard's bound. If you suspect that the denominator is much smaller than $\det M$, you may use `ZM_inv_ratlift`.

`GEN ZM_inv_ratlift(GEN M, GEN *pd)` if `M` is a `ZM`, return a primitive matrix H such that MH is d times the identity and set `*pd` to d. Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use `ZM_inv`.

`GEN SL2_inv_shallow(GEN M)` return the inverse of $M \in \mathrm{SL}_2(\mathbf{Z})$. Shallow function.

`GEN ZM_pseudoinv(GEN M, GEN *pv, GEN *pd)` if `M` is a nonempty `ZM`, let $v = [y, z]$ returned by `indexrank` and let M_1 be the corresponding square invertible matrix. Return a primitive left-inverse H such that HM_1 is d times the identity and set `*pd` to d. If `pv` is not `NULL`, set `*pv` to v. Shallow function.

`GEN ZM_gauss(GEN a, GEN b)` as `gauss`, where a and b coefficients are `t_INT`s.

`GEN ZM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 if the `ZM` x is the identity matrix, and 0 otherwise.

`int ZM_isdiagonal(GEN x)` return 1 if the `ZM` x is diagonal, and 0 otherwise.

`int ZM_isscalar(GEN x, GEN s)` given a `ZM` x and a `t_INT` s, return 1 if x is equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

`long ZC_is_ei(GEN x)` return i if the `ZC` x has 0 entries, but for a 1 at position i.

`int ZM_ishnf(GEN x)` return 1 if x is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for $j > i$, $x_{i,i} > x_{i,j} \geq 0$.

7.5.2 QM.

`GEN QM_charpoly_ZX(GEN M)` returns the characteristic polynomial (in variable 0) of the `QM` M, assuming that the result has integer coefficients.

`GEN QM_charpoly_ZX_bound(GEN M, long b)` as `QM_charpoly_ZX` assuming that the sup norm of the (integral) result is $\leq 2^b$.

`GEN QM_gauss(GEN a, GEN b)` as `gauss`, where a and b coefficients are `t_FRAC`s.

`GEN QM_gauss_i(GEN a, GEN b, long flag)` as `QM_gauss` if `flag` is 0. Else, no longer assume that a is left-invertible and return a solution of $Pax = Pb$ where P is a row-selection matrix such that $A = PaQ$ is square invertible of maximal rank, for some column-selection matrix Q; in particular, x is a solution of the original equation $ax = b$ if and only if a solution exists.

`GEN QM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN QM_inv(GEN M)` return the inverse of the `QM` M.

`long QM_rank(GEN x)` returns `matrank(x)`.

`GEN QM_image(GEN x)` returns an integral matrix with primitive columns generating the image of x.

`GEN QM_image_shallow(GEN A)` shallow version of the previous function.

`int QM_ishnf(GEN x)` return 1 if dx is in HNF form for some positive integer x.

7.5.3 Qevproj.

`GEN Qevproj_init(GEN M)` let M be a $n \times d$ `ZM` of maximal rank $d \leq n$, representing the basis of a $\mathbf{Q}$-subspace V of $\mathbf{Q}^n$. Return a projector on V, to be used by `Qevproj_apply`. The interface details may change in the future, but this function currently returns $[M, B, D, p]$, where p is a `t_VECSMALL` with d entries such that the submatrix $A =$ `rowpermute`(M, p) is invertible, B is a `ZM` and d a `t_INT` such that $AB = D\mathrm{Id}_d$.

`GEN Qevproj_apply(GEN T, GEN pro)` let T be an $n \times n$ `QM`, stabilizing a $\mathbf{Q}$-subspace $V \subset \mathbf{Q}^n$ of dimension d, and let `pro` be a projector on that subspace initialized by `Qevproj_init`(M). Return the $d \times d$ matrix representing $T_{|V}$ on the basis given by the columns of M.

`GEN Qevproj_apply_vecei(GEN T, GEN pro, long k)` as `Qevproj_apply`, return only the image of the k-th basis vector $M[k]$ (still on the basis given by the columns of M).

`GEN Qevproj_down(GEN T, GEN pro)` given a `ZC` (resp. a `ZM`) T representing an element (resp. a vector of elements) in the subspace V return a `QC` (resp. a `QM`) U such that $T = MU$.

7.5.4 zv, zm.

`GEN identity_zv(long n)` return the `t_VECSMALL` $[1, 2, \ldots, n]$.

`GEN random_zv(long n)` returns a random `zv` with n components.

`GEN zv_abs(GEN x)` return $[|x[1]|, \ldots, |x[n]|]$ as a `zv`.

`GEN zv_neg(GEN x)` return $-x$. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

`GEN zv_neg_inplace(GEN x)` negates x in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

`int zv_canon_inplace(GEN x)` return 1 if the first non-zero entry of x is positive (or x is 0); else return -1 and replace x by $-x$ in place by negating its entries. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

`GEN zm_zc_mul(GEN x, GEN y)`

`GEN zm_mul(GEN x, GEN y)`

`GEN zv_z_mul(GEN x, long n)` return nx. No check for overflow is done.

`long zv_content(GEN x)` returns the gcd of the entries of x.

`long zv_dotproduct(GEN x, GEN y)`

`long zv_prod(GEN x)` returns the product of all the components of `x` (assumes no overflow occurs).

`GEN zv_prod_Z(GEN x)` returns the product of all the components of `x`; consider all $x[i]$ as `ulong`s.

`long zv_sum(GEN x)` returns the sum of all the components of `x` (assumes no overflow occurs).

`long zv_sumpart(GEN v, long n)` returns the sum $v[1] + \ldots + v[n]$ (assumes no overflow occurs and $\texttt{lg}(v) > n$).

`int zv_cmp0(GEN x)` returns 1 if all entries of the `zv` x are 0, and 0 otherwise.

`int zv_equal(GEN x, GEN y)` returns 1 if the two `zv` are equal and 0 otherwise.

`int zv_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

`long zv_search(GEN L, long y)` look for y in the sorted `zv` L. Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN zv_copy(GEN x)` as `Flv_copy`.

`GEN zm_transpose(GEN x)` as `Flm_transpose`.

`GEN zm_copy(GEN x)` as `Flm_copy`.

`GEN zero_zm(long m, long n)` as `zero_Flm`.

`GEN zero_zv(long n)` as `zero_Flv`.

`GEN zm_row(GEN A, long x0)` as `Flm_row`.

`GEN zv_diagonal(GEN v)` return the square `zm` whose diagonal is given by the entries of v.

`GEN zm_permanent(GEN M)` return the permanent of M. The function assumes that the matrix is square of dimension $<$ `BITS_IN_LONG`.

`int zvV_equal(GEN x, GEN y)` returns 1 if the two `zvV` (vectors of `zv`) are equal and 0 otherwise.

7.5.5 ZMV / zmV (vectors of ZM/zm).

`int RgV_is_ZMV(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if its components are `ZM`, and 0 otherwise.

`GEN ZMV_to_zmV(GEN z)`

`GEN zmV_to_ZMV(GEN z)`

`GEN ZMV_to_FlmV(GEN z, ulong m)`

7.5.6 QC / QV, QM.

`GEN QM_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN QM_sqr(GEN x)` returns the square of `x` (assumed to be square).

`GEN QM_QC_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN QM_det(GEN M)` returns the determinant of M.

`GEN QM_ker(GEN x)` returns `matker(x)`.

7.5.7 RgC / RgV, RgM.

`RgC` and `RgV` routines assume the inputs are `VEC` or `COL` of the same dimension. `RgM` assume the inputs are `MAT` of compatible dimensions.

7.5.7.1 Matrix arithmetic.

`void RgM_dimensions(GEN x, long *m, long *n)` sets m, resp. n, to the number of rows, resp. columns of the `t_MAT` x.

`GEN RgC_add(GEN x, GEN y)` returns $x + y$ as a `t_COL`.

`GEN RgC_neg(GEN x)` returns $-x$ as a `t_COL`.

`GEN RgC_sub(GEN x, GEN y)` returns $x - y$ as a `t_COL`.

`GEN RgV_add(GEN x, GEN y)` returns $x + y$ as a `t_VEC`.

`GEN RgV_neg(GEN x)` returns $-x$ as a `t_VEC`.

`GEN RgV_sub(GEN x, GEN y)` returns $x - y$ as a `t_VEC`.

`GEN RgM_add(GEN x, GEN y)` return $x + y$.

`GEN RgM_neg(GEN x)` returns $-x$.

`GEN RgM_sub(GEN x, GEN y)` returns $x - y$.

`GEN RgM_Rg_add(GEN x, GEN y)` assuming x is a square matrix and y a scalar, returns the square matrix $x + y * \mathrm{Id}$.

`GEN RgM_Rg_add_shallow(GEN x, GEN y)` as `RgM_Rg_add` with much fewer copies. Shallow function.

`GEN RgM_Rg_sub(GEN x, GEN y)` assuming x is a square matrix and y a scalar, returns the square matrix $x - y * \mathrm{Id}$.

`GEN RgM_Rg_sub_shallow(GEN x, GEN y)` as `RgM_Rg_sub` with much fewer copies. Shallow function.

`GEN RgC_Rg_add(GEN x, GEN y)` assuming x is a nonempty column vector and y a scalar, returns the vector $[x_1 + y, x_2, \ldots, x_n]$.

`GEN RgC_Rg_sub(GEN x, GEN y)` assuming x is a nonempty column vector and y a scalar, returns the vector $[x_1 - y, x_2, \ldots, x_n]$.

`GEN Rg_RgC_sub(GEN a, GEN x)` assuming x is a nonempty column vector and a a scalar, returns the vector $[a - x_1, -x_2, \ldots, -x_n]$.

`GEN RgC_Rg_div(GEN x, GEN y)`

`GEN RgM_Rg_div(GEN x, GEN y)` returns x/y (y treated as a scalar).

`GEN RgC_Rg_mul(GEN x, GEN y)`

`GEN RgV_Rg_mul(GEN x, GEN y)`

`GEN RgM_Rg_mul(GEN x, GEN y)` returns $x \times y$ (y treated as a scalar).

`GEN RgV_RgC_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgV_RgM_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_RgC_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_RgX_mul(GEN x, GEN T)` returns $x \times y$, where y is `RgX_to_RgC`$(T, \texttt{lg}(x) - 1)$.

`GEN RgM_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_div(GEN a, GEN b)` returns $ab^{-}1$ or `NULL` if this turns out to be impossible.

`GEN RgM_ZM_mul(GEN x, GEN y)` returns $x \times y$ assuming that y is a `ZM`.

`GEN RgM_transmul(GEN x, GEN y)` returns $x^{\sim} \times y$.

`GEN RgM_multosym(GEN x, GEN y)` returns $x \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

`GEN RgM_transmultosym(GEN x, GEN y)` returns $x^{\sim} \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

`GEN RgMrow_RgC_mul(GEN x, GEN y, long i)` multiplies the i-th row of `RgM x` by the `RgC y` (seen as a column vector, assumed to have compatible dimensions). Assumes that x is nonempty and $0 < i < \texttt{lg}(\texttt{x}[1])$.

`GEN RgM_mulreal(GEN x, GEN y)` returns the real part of $x \times y$ (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`).

`GEN RgM_sqr(GEN x)` returns x^2.

`GEN RgC_RgV_mul(GEN x, GEN y)` returns $x \times y$ (the matrix $(x_i y_j)$).

`GEN RgC_RgV_mulrealsym(GEN x, GEN y)` returns the real part of $x \times y$ (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`), assuming the result is symmetric.

The following two functions are not well defined in general and only provided for convenience in specific cases:

`GEN RgC_RgM_mul(GEN x, GEN y)` returns $x \times y[1,]$ if y is a row matrix $1 \times n$, error otherwise.

`GEN RgM_RgV_mul(GEN x, GEN y)` returns $x \times y[,1]$ if y is a column matrix $n \times 1$, error otherwise.

`GEN RgM_powers(GEN x, long n)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^n]$ as a `t_VEC` of `RgM`s.

`GEN RgV_sum(GEN v)` sum of the entries of v

`GEN RgV_prod(GEN v)` product of the entries of v, using a divide and conquer strategy

`GEN RgV_sumpart(GEN v, long n)` returns the sum $v[1] + \ldots + v[n]$ (assumes that $\mathtt{lg}(v) > n$).

`GEN RgV_sumpart2(GEN v, long m, long n)` returns the sum $v[m] + \ldots + v[n]$ (assumes that $\mathtt{lg}(v) > n$ and $m > 0$). Returns `gen_0` when $m > n$.

`GEN RgM_sumcol(GEN v)` returns a `t_COL`, sum of the columns of the `t_MAT` v.

`GEN RgV_dotproduct(GEN x, GEN y)` returns the scalar product of x and y

`GEN RgV_dotsquare(GEN x)` returns the scalar product of x with itself.

`GEN RgV_kill0(GEN v)` returns a shallow copy of v where entries matched by `gequal0` are replaced by `NULL`. The return value is not a valid `GEN` and must be handled specially. The idea is to pre-treat a vector of coefficients to speed up later linear combinations or scalar products.

`GEN gram_matrix(GEN v)` returns the Gram matrix $(v_i \cdot v_j)$ attached to the entries of v (matrix, or vector of vectors).

`GEN RgV_polint(GEN X, GEN Y, long v)` X and Y being two vectors of the same length, returns the polynomial T in variable v such that $T(X[i]) = Y[i]$ for all i. The special case $X =$ `NULL` corresponds to $X = [1, 2, \ldots, n]$, where n is the length of Y. This is the function underlying `polint` for formal interpolation.

`GEN polintspec(GEN X, GEN Y, GEN t, long n, long *pe)` return $P(t)$ where P is the Lagrange interpolation polynomial attached to the n points $(X[0], Y[0]), \ldots, (X[n-1], Y[n-1])$. If `pe` is not `NULL` and t is a complex numeric value, `*pe` contains an error estimate for the returned value (Neville's algorithm, see `polinterpolate`). In extrapolation algorithms, e.g., Romberg integration, this function is usually called on actual `GEN` vectors with offsets: $x+k$ and $y+k$ so as to interpolate on $x[k..k+n-1]$ without having to use `vecslice`. This is the function underlying `polint` for numerical interpolation.

`GEN polint_i(GEN X, GEN Y, GEN t, long *pe)` as `polintspec`, where X and Y are actual `GEN` vectors.

`GEN vandermondeinverse(GEN r, GEN T, GEN d, GEN V)` Given a vector r of n scalars and the `t_POL` $T = \prod_{i=1}^{n}(X - r_i)$, return dM^{-1}, where $M = (r_i^{j-1})_{1 \le i,j \le n}$ is the van der Monde matrix; V is `NULL` or a vector containing the $T'(r_i)$, as returned by `vandermodeinverseinit`. The demonimator d may be set to `NULL` (handled as 1). If c is the k-th column of the result, it is essentially d times the k-th Lagrange interpolation polynomial: we have $\sum_j c_j r_i^{j-1} = d\delta_{i=k}$.

This function underlies `RgV_polint` when the base field is not $\mathbf{Z}/p\mathbf{Z}$: it uses $O(n^2)$ scalar operations and is asymptotically slower than variants using multi-evaluation such as `FpV_polint`; it is also accurate over inexact fields.

A technical variant allows V from `vandermodeinverseinit_real` (complex roots up to conjugation). Then the output need no longer be a square matrix: only the columns corresponding to the given roots representative are included. (The others are implied, being their conjugates.) This allows to reconstruct an algebraic number in $\mathbf{Q}[X]/(T)$ given $r_1 + r_2$ real and (non-conjugate) complex embeddings only, instead of $r_1 + 2r_2 = \deg T$.

`GEN vandermondeinverseinit(GEN r)` Given a vector r of n scalars, let T be the `t_POL` $T = \prod_{j=1}^{n}(X - r_j)$. This function returns the $T'(r_i)$ computed stably via products of difference: the i-th entry is $T'(r_i) = \prod_{j \neq i}(r_i - r_j)$. It is asymptotically slow (uses $O(n^2)$ scalar operations, where multi-evaluation achieves quasi-linear running time) but allows accurate computation at low accuracies when T has large complex coefficients.

`GEN vandermondeinverseinit_real(GEN R, long r1)` This variant improves on `vandermondeinverse` in the important special case where the roots are complex and T has real coefficients, i.e., the set of roots is stable under complex conjugation. The argument r is replaced by a set R of representatives modulo conjugation: the first r_1 elements of R are real and the remaining r_2 are representatives of pairs of complex conjugates. This returns the $T'(R_i)$, from which the $T'(r_i)$ can be reconstructed by adding appropriate conjugates.

7.5.7.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

`int RgV_isscalar(GEN x)` return 1 if all the entries of x are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis $1, T, \dots, T^{n-1}$, or on `nf.zk` (whose first element is 1).

`int QV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `QV` (`t_INT` and `t_FRAC` entries only).

`int ZV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `ZV` (`t_INT` entries only).

`int RgM_isscalar(GEN x, GEN s)` return 1 if x is the scalar matrix equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

`int RgM_isidentity(GEN x)` return 1 if the `t_MAT` x is the identity matrix, and 0 otherwise.

`int RgM_isdiagonal(GEN x)` return 1 if the `t_MAT` x is a diagonal matrix, and 0 otherwise.

`long RgC_is_ei(GEN x)` return i if the `t_COL` x has 0 entries, but for a 1 at position i.

`int RgM_is_ZM(GEN x)` return 1 if the `t_MAT` x has only `t_INT` coefficients, and 0 otherwise.

`long qfiseven(GEN M)` return 1 if the square symmetric typZM x is an even quadratic form (all diagonal coefficients are even), and 0 otherwise.

`int RgM_is_QM(GEN x)` return 1 if the `t_MAT` x has only `t_INT` or `t_FRAC` coefficients, and 0 otherwise.

`long RgV_isin(GEN v, GEN x)` return the first index i such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted.

`long RgV_isin_i(GEN v, GEN x, long n)` return the first index i *leqn* such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted. Assume that $n < \mathtt{lg}(\mathtt{v})$.

`GEN RgM_diagonal(GEN m)` returns the diagonal of m as a `t_VEC`.

`GEN RgM_diagonal_shallow(GEN m)` shallow version of `RgM_diagonal`

7.5.7.3 Conversion to floating point entries.

`GEN RgC_gtofp(GEN x, GEN prec)` returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgV_gtofp(GEN x, GEN prec)` returns the `t_VEC` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_gtomp(GEN x, long prec)` returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgC_gtofp(x, prec) )
```

`GEN RgM_gtofp(GEN x, GEN prec)` returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_gtomp(GEN x, long prec)` returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

7.5.7.4 Linear algebra, linear systems.

`GEN RgM_inv(GEN a)` returns a left inverse of a (which needs not be square), or `NULL` if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

`GEN RgM_inv_upper(GEN a)` as `RgM_inv`, assuming that a is a nonempty invertible upper triangular matrix, hence a little faster. The diagonal coefficients of the inverse matrix are computed using `ginv(gcoeff(a,i,i))`; in particular, they remain rational when the original diagonal coefficients were rational.

`GEN RgM_RgC_invimage(GEN A, GEN B)` returns a `t_COL` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_invimage(GEN A, GEN B)` returns a `t_MAT` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_Hadamard(GEN a)` returns a upper bound for the absolute value of $\det(a)$. The bound is a `t_INT`.

`GEN RgM_solve(GEN a, GEN b)` returns $a^{-1}b$ where a is a square `t_MAT` and b is a `t_COL` or `t_MAT`. Returns `NULL` if a^{-1} cannot be computed, see `RgM_inv`.

If $b =$ `NULL`, the matrix a need no longer be square, and we strive to return a left inverse for a (`NULL` if it does not exist).

`GEN RgM_solve_realimag(GEN M, GEN b)` M being a `t_MAT` with r_1+r_2 rows and r_1+2r_2 columns, y a `t_COL` or `t_MAT` such that the equation $Mx = y$ makes sense, returns x under the following simplifying assumptions: the first r_1 rows of M and y are real (the r_2 others are complex), and x is real. This is stabler and faster than calling `RgM_solve`(M, b) over $\mathbf{C}$. In most applications, M approximates the complex embeddings of an integer basis in a number field, and x is actually rational.

`GEN split_realimag(GEN x, long r1, long r2)` x is a `t_COL` or `t_MAT` with r_1+r_2 rows, whose first r_1 rows have real entries (the r_2 others are complex). Return an object of the same type as x and r_1+2r_2 rows, such that the first r_1+r_2 rows contain the real part of x, and the r_2 following ones contain the imaginary part of the last r_2 rows of x. Called by `RgM_solve_realimag`.

`GEN RgM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`GEN Frobeniusform(GEN V, long n)` given the vector V of elementary divisors for $M-x\mathrm{Id}$, where M is an $n\times n$ square matrix. Returns the Frobenius form of M.

`int RgM_QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` QR-decomposition of a square invertible `t_MAT` x with real coefficients. Sets `*pB` to the vector of squared lengths of the (Gram-Schmidt) orthogonalized vectors $x[i]^*$, `*pL` to the Gram-Schmidt coefficients and `*pQ` to a vector of successive Householder transforms. If R denotes the transpose of L and Q is the result of applying `*pQ` to the identity matrix, then $x=QR$ is the QR decomposition of x: Q is orthogonal and R is upper triangular with positive diagonal, both matrices having `t_INT` or `t_REAL` coefficients. Abort and returns 0 is x if not invertible or we hit a precision problem (none of B, Q, L should be accessed in that case), and 1 otherwise.

`int QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` as `RgM_QR_init`, assuming further that x has `t_INT` or `t_REAL` coefficients.

`GEN R_from_QR(GEN x, long prec)` assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, return the upper triangular R from the QR docomposition of x. Not memory clean. If the matrix is not known to have `t_INT` or `t_REAL` coefficients, apply `RgM_gtomp` first.

`GEN gaussred_from_QR(GEN x, long prec)` assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, returns `qfgaussred(x~* x)`; this is essentially the upper triangular R matrix from the QR decomposition of x, renormalized to accomodate `qfgaussred` conventions. Not stack-clean.

`GEN RgM_gram_schmidt(GEN e, GEN *ptB)` naive (unstable) Gram-Schmidt orthogonalization of the basis (e_i) given by the columns of `t_MAT` e. Return the e_i^* (as columns of a `t_MAT`) and set `*ptB` to the vector of squared lengths $|e_i^*|^2$.

`GEN RgM_Babai(GEN M, GEN y)` given a `t_MAT` M of maximal rank n and a `t_COL` y of the same dimension, apply Babai's nearest plane algorithm to return an *integral* x such that $y-Mx$ has small L_2 norm. This yields an approximate solution to the closest vector problem: if M is LLL-reduced, then

$$||y-Mx||_2 \le 2(2/\sqrt{3})^n||y-MX||_2$$

for all $X\in\mathbf{Z}^n$.

`GEN RgM_Cholesky(GEN M)` given a square symmetric `t_MAT` M, return R such that ${}^tRR=M$, or `NULL` is no solution is found (M is not positive or was given with insufficient accuracy).

7.5.8 ZG.

Let G be a multiplicative group with neutral element 1_G whose multiplication is supported by `gmul` and where equality test is performed using `gidentical`, e.g. a matrix group. The following routines implement basic computations in the group algebra $\mathbf{Z}[G]$. All of them are shallow for efficiency reasons. A `ZG` is either

• a `t_INT` n, representing $n[1_G]$

• or a "factorization matrix" with two columns $[g, e]$: the first one contains group elements, sorted according to `cmp_universal`, and the second one contains integer "exponents", representing $\sum e_i[g_i]$.

Note that `to_famat` and `to_famat_shallow`(g, e) allow to build the `ZG` $e[g]$ from $e \in \mathbf{Z}$ and $g \in G$.

`GEN ZG_normalize(GEN x)` given a `t_INT` x or a factorization matrix *without* assuming that the first column is properly sorted. Return a valid (sorted) `ZG`. Shallow function.

`GEN ZG_add(GEN x, GEN y)` return $x + y$; shallow function.

`GEN ZG_neg(GEN x)` return $-x$; shallow function.

`GEN ZG_sub(GEN x, GEN y)` return $x - y$; shallow function.

`GEN ZG_mul(GEN x, GEN y)` return xy; shallow function.

`GEN ZG_G_mul(GEN x, GEN y)` given a `ZG` x and $y \in G$, return xy; shallow function.

`GEN G_ZG_mul(GEN x, GEN y)` given a `ZG` y and $x \in G$, return xy; shallow function.

`GEN ZG_Z_mul(GEN x, GEN n)` given a `ZG` x and $y \in \mathbf{Z}$, return xy; shallow function.

`GEN ZGC_G_mul(GEN v, GEN x)` given v a vector of `ZG` and $x \in G$ return the vector (with the same type as v with entries $v[i] \cdot x$. Shallow function.

`void ZGC_G_mul_inplace(GEN v, GEN x)` as `ZGC_G_mul`, modifying v in place.

`GEN ZGC_Z_mul(GEN v, GEN n)` given v a vector of `ZG` and $n \in Z$ return the vector (with the same type as v with entries $n \cdot v[i]$. Shallow function.

`GEN G_ZGC_mul(GEN x, GEN v)` given v a vector of `ZG` and $x \in G$ return the vector of $x \cdot v[i]$. Shallow function.

`GEN ZGCs_add(GEN x, GEN y)` add two sparse vectors of `ZG` elements (see Sparse linear algebra below).

7.5.9 Sparse and blackbox linear algebra.

A sparse column `zCs` v is a `t_COL` with two components C and E which are `t_VECSMALL` of the same length, representing $\sum_i E[i] * e_{C[i]}$, where (e_j) is the canonical basis. A sparse matrix (`zMs`) is a `t_VEC` of `zCs`.

`FpCs` and `FpMs` are identical to the above, but $E[i]$ is now interpreted as a *signed* C long integer representing an element of $\mathbf{F}_p$. This is important since p can be so large that $p + E[i]$ would not fit in a C long.

`RgCs` and `RgMs` are similar, except that the type of the components of E is now unspecified. Functions handling those later objects must not depend on the type of those components.

`F2Ms` are `t_VEC` of `F2Cs`. `F2Cs` are `t_VECSMALL` whoses entries are the nonzero coefficients (1).

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument `nbrow` which is the number of rows of the corresponding column/matrix in dense representation.

`GEN F2Ms_to_F2m(GEN M, long nbrow)` convert a `F2m` to a `F2Ms`.

`GEN F2m_to_F2Ms(GEN M)` convert a `F2m` to a `F2Ms`.

`GEN zCs_to_ZC(GEN C, long nbrow)` convert the sparse vector C to a dense `ZC` of dimension `nbrow`.

`GEN zMs_to_ZM(GEN M, long nbrow)` convert the sparse matrix M to a dense `ZM` whose columns have dimension `nbrow`.

`GEN zMs_compress(GEN M, long nbrow)` Renumber the rows to use consecutive integers starting at 1, skipping unused ones. Return $[L, p]$ where L is the new `zMs` and p is a `t_VECSMALL` giving the correspondance from the new column numbers to the original ones. the new number of rows is the length of p.

`GEN FpMs_FpC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M (over $\mathbf{F}_p$) by the `FpC` B. The result is an `FpC`, i.e. a dense vector.

`GEN zMs_ZC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M by the `ZC` B (over $\mathbf{Z}$). The result is an `ZC`, i.e. a dense vector. (Not stack clean).

`GEN FpV_FpMs_mul(GEN B, GEN M, GEN p)` multiply the `FpV` B by the sparse matrix M (over $\mathbf{F}_p$). The result is an `FpV`, i.e. a dense vector.

`GEN ZV_zMs_mul(GEN B, GEN M, GEN p)` multiply the `FpV` B (over $\mathbf{Z}$) by the sparse matrix M. The result is an `ZV`, i.e. a dense vector.

`void RgMs_structelim(GEN M, long nbrow, GEN A, GEN *p_col, GEN *p_row)` M being a `RgMs` with `nbrow` rows, A being a list of row indices, perform structured elimination on M by removing some rows and columns until the number of effectively present rows is equal to the number of columns. The result is stored in two `t_VECSMALL`s, `*p_col` and `*p_row`: `*p_col` is a map from the new columns indices to the old one. `*p_row` is a map from the old rows indices to the new one (0 if removed).

`GEN F2Ms_colelim(GEN M, long nbrow)` returns some subset of the columns of M as a `t_VECSMALL` of indices, selected such that the dimension of the kernel of the matrix is preserved. The subset is not guaranteed to be minimal.

`GEN F2Ms_ker(GEN M, long nbrow)` returns some kernel vectors of M using block Lanczos algorithm.

`GEN FpMs_leftkernel_elt(GEN M, long nbrow, GEN p)` M being a sparse matrix over $\mathbf{F}_p$, return a nonzero `FpV` X such that XM components are almost all 0.

`GEN FpMs_FpCs_solve(GEN M, GEN B, long nbrow, GEN p)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{F}_p$. Return either a solution as a `t_COL` (dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if B is not in the image of M).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that p is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{Z}/p^e\mathbf{Z}$. Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if B is not in the image of M).

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation $f(X) = B$ over $\mathbf{F}_p$, where B is a `FpV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`, or `NULL` (can happen if B is not in the image of M).

`GEN gen_ZpM_Dixon_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation $f(X) = B$ over $\mathbf{Z}/p^e\mathbf{Z}$, where B is a `ZV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`, or `NULL` (can happen if B is not in the image of M).

`GEN gen_ZpM_Dixon(GEN F, GEN B, GEN q, GEN p, long N, void *E, GEN lin(void *, GEN F, GEN d, GEN q), GEN inv(void *E, GEN d))` Let F be a square invertible `ZpM` of dimension k representing an automorphism f of $(\mathbf{Z}_p)^k$, let B be an element of $(\mathbf{Z}_p)^k$ and let $q = p^N$. Return $y \in (\mathbf{Z}_p)^k$ such that $f(y) = B \pmod{p^N}$ assuming the following holds for $n \leq N$:

- $\texttt{lin}(E, \texttt{FpM\_red}(F, p^n), z, p^n) \equiv f(z) \pmod{p^n}$
- $f(\texttt{inv}(E, z)) \equiv z \pmod{p}$

The argument F is redundant; including it allows the function to reduce it to the required p-adic precision when evaluating f modulo p^n.

7.5.10 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of x using a very slow algorithm. Use `image` instead.

7.6 Integral, rational and generic polynomial arithmetic.

7.6.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `ZX` (s should point to the name of the caller).

`GEN ZX_copy(GEN x, GEN p)` returns a copy of `x`.

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in x.

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x.

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x. Shallow function.

`GEN ZX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`int ZX_equal(GEN x, GEN y)` returns 1 if the two `ZX` have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

`int ZX_equal1(GEN x)` returns 1 if the `ZX` x is equal to 1 and 0 otherwise.

`int ZX_is_monic(GEN x)` returns 1 if the `ZX` x is monic and 0 otherwise. The zero polynomial considered not monic.

`GEN ZX_add(GEN x, GEN y)` adds `x` and `y`.

`GEN ZX_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN ZX_neg(GEN x)` returns $-$`x`.

`GEN ZX_Z_add(GEN x, GEN y)` adds the integer `y` to the `ZX` `x`.

`GEN ZX_Z_add_shallow(GEN x, GEN y)` shallow version of `ZX_Z_add`.

`GEN ZX_Z_sub(GEN x, GEN y)` subtracts the integer `y` to the `ZX` `x`.

`GEN Z_ZX_sub(GEN x, GEN y)` subtracts the `ZX` `y` to the integer `x`.

`GEN ZX_Z_mul(GEN x, GEN y)` multiplies the `ZX` `x` by the integer `y`.

`GEN ZX_mulu(GEN x, ulong y)` multiplies `x` by the integer `y`.

`GEN ZX_shifti(GEN x, long n)` shifts all coefficients of `x` by n bits, which can be negative.

`GEN ZX_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZX_divuexact(GEN x, ulong y)` returns x/y assuming all divisions are exact.

`GEN ZX_remi2n(GEN x, long n)` reduces all coefficients of `x` to n bits, using `remi2n`.

`GEN ZX_mul(GEN x, GEN y)` multiplies `x` and `y`.

`GEN ZX_sqr(GEN x, GEN p)` returns `x`^{2}.

`GEN ZX_mulspec(GEN a, GEN b, long na, long nb)`. Internal routine: `a` and `b` are arrays of coefficients representing polynomials $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$ and $\sum_{i=0}^{\mathtt{nb}-1} \mathtt{b}[i]X^i$. Returns their product (as a true `GEN`) in variable 0.

`GEN ZX_sqrspec(GEN a, long na)`. Internal routine: `a` is an array of coefficients representing polynomial $\sum_{i=0}^{\mathtt{na}-1} \mathtt{a}[i]X^i$. Return its square (as a true `GEN`) in variable 0.

`GEN ZX_rem(GEN x, GEN y)` returns the remainder of the Euclidean division of x mod y. Assume that x, y are two `ZX` and that y is monic.

`GEN ZX_mod_Xnm1(GEN T, ulong n)` return T modulo $X^n - 1$). Shallow function.

`GEN ZX_div_by_X_1(GEN T, GEN *r)` return the quotient of T by $X - 1$. If r is not `NULL` set it to $T(1)$.

`GEN ZX_digits(GEN x, GEN B)` returns a vector of `ZX` $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$. Assume that B is monic.

`GEN ZXV_ZX_fromdigits(GEN v, GEN B)` where $v = [c_0, \dots, c_n]$ is a vector of `ZX`, returns $\sum_{i=0}^{n} c_i B^i$.

`GEN ZX_gcd(GEN x, GEN y)` returns a gcd of the `ZX` x and y.

`GEN ZX_gcd_all(GEN x, GEN y, GEN *pX)` returns a gcd d of x and y. If `pX` is not `NULL`, set $*$`pX` to the primitive part of x/d. If x and y have coprime leading terms, then d is monic and `*pX` is `Q_primpart`(x) divided by d. Not stack-clean.

`GEN ZX_radical(GEN x)` returns the largest squarefree primitive divisor of the `ZX` x. Not stack-clean.

`GEN ZX_content(GEN x)` returns the content of the `ZX` x.

`long ZX_val(GEN P)` as `RgX_val`, but assumes `P` has `t_INT` coefficients.

`long ZX_valrem(GEN P, GEN *z)` as `RgX_valrem`, but assumes `P` has `t_INT` coefficients.

`GEN ZX_to_monic(GEN q GEN *L)` given q a nonzero `ZX`, returns a monic integral polynomial Q such that $Q(x) = Cq(x/L)$, for some rational C and positive integer $L > 0$. If `L` is not `NULL`, set `*L` to L; if $L = 1$, `*L` is set to `gen_1`. Shallow function.

`GEN ZX_primitive_to_monic(GEN q, GEN *L)` as `ZX_to_monic` except q is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if q is not primitive (L larger than necessary), but remains correct. Shallow function.

`GEN ZX_Z_normalize(GEN q, GEN *L)` a restricted version of `ZX_primitive_to_monic`, where q is a *monic* `ZX` of degree > 0. Finds the largest integer $L > 0$ such that $Q(X) := L^{-\deg q} q(Lx)$ is integral and return Q; this is not well-defined if q is a monomial, in that case, set $L = 1$ and $Q = q$. If `L` is not `NULL`, set `*L` to L. Shallow function.

`GEN ZX_Q_normalize(GEN q, GEN *L)` a variant of `ZX_Z_normalize` where $L > 0$ is allowed to be rational, the monic $Q \in \mathbf{Z}[X]$ has possibly smaller coefficients. Shallow function.

`GEN ZX_Q_mul(GEN x, GEN y)` returns $x * y$, where y is a rational number and the resulting `t_POL` has rational entries.

`long ZX_deflate_order(GEN P)` given a nonconstant `ZX` P, returns the largest exponent d such that P is of the form $P(x^d)$.

`long ZX_deflate_max(GEN P, long *d)`. Given a nonconstant polynomial with integer coefficients P, sets `d` to `ZX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function.

`GEN ZX_rescale(GEN P, GEN h)` returns $h^{\deg(P)} P(x/h)$. `P` is a `ZX` and `h` is a nonzero integer. Shallow function.

`GEN ZX_rescale2n(GEN P, long n)` returns $2^{n \deg(P)} P(x >> n)$ where `P` is a `ZX`.

`GEN ZX_rescale_lt(GEN P)` returns the monic integral polynomial $h^{\deg(P)-1}P(x/h)$, where `P` is a nonzero `ZX` and `h` is its leading coefficient. Shallow function.

`GEN ZX_Z_translate(GEN P, GEN c)` assume P is a `ZX` and c an integer. Returns $P(X+c)$ (optimized for $c = \pm 1$).

`GEN ZX_affine(GEN P, GEN a, GEN b)` P is a `ZX`, a and b are `t_INT`. Return $P(aX+b)$ (optimized for $b = \pm 1$). Not memory clean.

`GEN ZX_Z_eval(GEN P, GEN x)` evaluate the `ZX` P at the integer x.

`GEN ZX_unscale(GEN P, GEN h)` given a `ZX` P and a `t_INT` h, returns $P(hx)$. Assume $h \neq 0$. Not stack-clean.

`GEN ZX_z_unscale(GEN P, long h)` given a `ZX` P, returns $P(hx)$. Assume $h \neq 0$. Not stack-clean.

`GEN ZX_unscale2n(GEN P, long n)` given a `ZX` P, returns $P(x << n)$. Not stack-clean.

`GEN ZX_unscale_div(GEN P, GEN h)` given a `ZX` P and a `t_INT` h such that $h \mid P(0)$, returns $P(hx)/h$. Not stack-clean.

`GEN ZX_unscale_divpow(GEN P, GEN h, long k)` given a `ZX` P, a `t_INT` h and $k > 0$, returns $P(hx)/h^k$ assuming the result has integral coefficients. Not stack-clean.

`GEN ZX_eval1(GEN P)` returns the integer $P(1)$.

`GEN ZX_graeffe(GEN p)` returns the Graeffe transform of p, i.e. the `ZX` q such that $p(x)p(-x) = q(x^2)$.

`GEN ZX_deriv(GEN x)` returns the derivative of `x`.

`GEN ZX_resultant(GEN A, GEN B)` returns the resultant of the `ZX` `A` and `B`.

`GEN ZX_disc(GEN T)` returns the discriminant of the `ZX` `T`.

`GEN ZX_factor(GEN T)` returns the factorization of the primitive part of `T` over $\mathbf{Q}[X]$ (the content is lost).

`int ZX_is_squarefree(GEN T)` returns 1 if the `ZX` `T` is squarefree, 0 otherwise.

`long ZX_is_irred(GEN T)` returns 1 it `T` is irreducible, and 0 otherwise.

`GEN ZX_squff(GEN T, GEN *E)` write the primitive part of the nonzero `ZX` T as a product $\prod T_i^{e_i}$ with the $e_1 < e_2 < \cdots$ all distinct and the T_i primitive and pairwise coprime. Return the vector of the T_i, and set `*E` to the vector of the e_i, as a `t_VECSMALL`. For efficiency, powers of x should have been removed from T using `ZX_valrem`, but the result is also correct if not. Not stack-clean.

`GEN ZX_Uspensky(GEN P, GEN ab, long flag, long bitprec)` let `P` be a `ZX` polynomial whose real roots are simple and `bitprec` is the relative precision in bits. For efficiency reasons, `P` should not only have simple real roots but actually be primitive and squarefree, but the routine neither checks nor enforces this, and it returns correct results in this case as well.

- If `flag` is 0 returns a list of intervals that isolate the real roots of `P`. The return value is a column of elements which are either vectors `[a,b]` of rational numbers meaning that there is a single nonrational root in the open interval `(a,b)` or elements `x0` such that `x0` is a rational root of `P`. Beware that the limits of the open intervals can be roots of the polynomial.
- If `flag` is 1 returns an approximation of the real roots of `P`.
- If `flag` is 2 returns the number of roots.

The argument `ab` specify the interval in which the roots are searched. The default interval is $(-\infty, \infty)$. If `ab` is an integer or fraction a then the interval is $[a, \infty)$. If `ab` is a vector $[a, b]$, where `t_INT`, `t_FRAC` or `t_INFINITY` are allowed for a and b, the interval is $[a, b]$.

`long ZX_sturm(GEN P)` number of real roots of the nonconstant squarefree `ZX` P. For efficiency, it is advised to make P primitive first. If P is already known to be irreducible, use `ZX_sturm_irred` for a small speed up.

`long ZX_sturmpart(GEN P, GEN ab)` number of real roots of the nonconstant squarefree `ZX` P in the interval specified by `ab`: either `NULL` (no restriction) or a `t_VEC` $[a, b]$ with two real components (of type `t_INT`, `t_FRAC` or `t_INFINITY`). For efficiency, it is advised to make P primitive first.

`long ZX_sturm_irred(GEN P)` number of real roots of the irreducible `ZX` P. For efficiency, it is advised to make P primitive first.

`long ZX_realroots(GEN P, long prec)` real roots of the nonconstant squarefree `ZX` P, to precision prec. For efficiency, it is advised to make P primitive first. If P is known to be irreducible, `ZX_realroots_irred` will be faster.

`long ZX_realroots_irred(GEN P, long prec)` real roots of the irreducible `ZX` P, to precision prec. For efficiency, it is advised to make P primitive first.

7.6.2 Resultants.

`GEN ZX_ZXY_resultant(GEN A, GEN B)` under the assumption that `A` in $\mathbf{Z}[Y]$, `B` in $\mathbf{Q}[Y][X]$, and $R = \mathrm{Res}_Y(A, B) \in \mathbf{Z}[X]$, returns the resultant R.

`GEN ZXX_resultant(GEN A, GEN B)` returns the resultant with respect to the main variable.

`GEN ZX_composedsum(GEN P, GEN Q)` if $P = a\prod_i(x - p_i)$ and $Q = b\prod_j(x - q_j)$ in some suitable algebraic extension, return $ab\prod_{i,j}(x - (p_i + q_j))$.

`GEN ZX_compositum(GEN A, GEN B, long *lambda)` given two irreducible `ZX`, returns an irreducible `ZX` C defining their compositum and set `lambda` to a small integer k such that if α is a root of A and β is a root of B, then $k\alpha + \beta$ is a root of C.

`GEN ZX_ZXY_rnfequation(GEN A, GEN B, long *lambda)`, assume `A` in $\mathbf{Z}[Y]$, `B` in $\mathbf{Q}[Y][X]$, and $R = \mathrm{Res}_Y(A, B) \in \mathbf{Z}[X]$. If `lambda = NULL`, returns R as in `ZY_ZXY_resultant`. Otherwise, `lambda` must point to some integer, e.g. 0 which is used as a seed. The function then finds a small $\lambda \in \mathbf{Z}$ (starting from `*lambda`) such that $R_\lambda(X) := \mathrm{Res}_Y(A, B(X + \lambda Y))$ is squarefree, resets `*lambda` to the chosen value and returns R_λ.

7.6.3 ZXV.

`GEN ZXV_equal(GEN x, GEN y)` returns 1 if the two vectors of `ZX` are equal, as per `ZX_equal` (variables are not checked to be equal) and 0 otherwise.

`GEN ZXV_Z_mul(GEN x, GEN y)` multiplies the vector of `ZX` `x` by the integer `y`.

`GEN ZXV_remi2n(GEN x, long n)` applies `ZX_remi2n` to all coefficients of `x`.

`GEN ZXV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming x and y have `ZX` entries.

7.6.4 ZXT.

`GEN ZXT_remi2n(GEN x, long n)` applies `ZX_remi2n` to all leaves of the tree `x`.

7.6.5 ZXQ.

`GEN ZXQ_mul(GEN x, GEN y, GEN T)` returns $x * y$ mod T, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_sqr(GEN x, GEN T)` returns x^2 mod T, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_powu(GEN x, ulong n, GEN T)` returns x^n mod T, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_powers(GEN x, long n, GEN T)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ mod T as a `t_VEC`, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQ_charpoly(GEN A, GEN T, long v)`: let `T` and `A` be `ZX`s, returns the characteristic polynomial of `Mod(A, T)`. More generally, `A` is allowed to be a `QX`, hence possibly has rational coefficients, *assuming* the result is a `ZX`, i.e. the algebraic number `Mod(A,T)` is integral over `Z`.

`GEN ZXQ_minpoly(GEN A, GEN T, long d, ulong bound)` let `T` and `A` be `ZX`s, returns the minimal polynomial of `Mod(A, T)` assuming it has degree d and its coefficients are less than 2^{bound}. More generally, `A` is allowed to be a `QX`, hence possibly has rational coefficients, *assuming* the result is a `ZX`, i.e. the algebraic number `Mod(A,T)` is integral over `Z`.

7.6.6 ZXn.

`GEN ZXn_mul(GEN x, GEN y, long n)` return $xy \pmod{X^n}$.

`GEN ZXn_sqr(GEN x, long n)` return $x^2 \pmod{X^n}$.

`GEN eta_ZXn(long r, long n)` return $\eta(X^r) = \prod_{i>0}(1 - X^{ri}) \pmod{X^n}$, $r > 0$.

`GEN eta_product_ZXn(GEN DR, long n)`: `DR` $= [D, R]$ being a vector with two `t_VECSMALL` components, return $\prod_i \eta(X^{d_i})^{r_i}$. Shallow function.

7.6.7 ZXQM.

`ZXQM` are matrices of `ZXQ`. All entries must be integers or polynomials of degree strictly less than the degree of T.

`GEN ZXQM_mul(GEN x, GEN y, GEN T)` returns $x * y$ mod T, assuming that all inputs are `ZX`s and that T is monic.

`GEN ZXQM_sqr(GEN x, GEN T)` returns x^2 mod T, assuming that all inputs are `ZX`s and that T is monic.

7.6.8 ZXQX.

`GEN ZXQX_mul(GEN x, GEN y, GEN T)` returns $x * y$, assuming that all inputs are `ZXQX`s and that T is monic.

`GEN ZXQX_ZXQ_mul(GEN x, GEN y, GEN T)` returns $x * y$, assuming that x is a `ZXQX`, y is a `ZXQ` and T is monic.

`GEN ZXQX_sqr(GEN x, GEN T)` returns x^2, assuming that all inputs are `ZXQX`s and that T is monic.

`GEN ZXQX_gcd(GEN x, GEN y, GEN T)` returns the gcd of x and y, assuming that all inputs are `ZXQX`s and that T is monic.

`GEN ZXQXQ_minpoly(GEN A, GEN B, GEN T, long d, ulong bound)` let `B` and `A` be `ZXQX`s mod T, returns the minimal polynomial of `Mod(A, B)` assuming it has degree d and its coefficients are less than 2^{bound}. More generally, `A` is allowed to be a `QXQX`, hence possibly has rational coefficients, *assuming* the result is a `ZXQX`, i.e. the algebraic number `Mod(A,B)` is integral mod T.

7.6.9 ZXX.

`void RgX_check_ZXX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it one of its coefficients is not an integer or a `ZX` (s should point to the name of the caller).

`GEN ZXX_renormalize(GEN x, long l)`, as `normalizepol`, where `l = lg(x)`, in place.

`long ZXX_max_lg(GEN x)` returns the effective length of the longest component in x; assume all coefficients are `t_INT` or `ZX`s.

`GEN ZXX_evalx0(GEN P)` returns $P(X, 0)$.

`GEN ZXX_Z_mul(GEN x, GEN y)` returns xy.

`GEN ZXX_shifti(GEN x, long n)` shifts all coefficients of `x` by n bits, which can be negative.

`GEN ZXX_Q_mul(GEN x, GEN y)` returns $x * y$, where y is a rational number and the resulting `t_POL` has rational entries.

`GEN ZXX_Z_add_shallow(GEN x, GEN y)` returns $x + y$. Shallow function.

`GEN ZXX_Z_divexact(GEN x, GEN y)` returns x/y assuming all integer divisions are exact.

`GEN Kronecker_to_ZXX(GEN z, long n, long v)` recover $P(X, Y)$ from its Kronecker form $P(X, X^{2n-1})$ (see `RgXX_to_Kronecker`), v is the variable number corresponding to Y. Shallow function.

`GEN Kronecker_to_ZXQX(GEN z, GEN T)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{Z}[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P (see `RgXX_to_Kronecker`), this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t)$ mod $(T(X))$, $\deg_X Q < n$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN ZXX_mul_Kronecker(GEN P, GEN Q, long n)` return `ZX_mul` applied to the Kronecker forms $P(X, X^{2n-1})$ and $Q(X, X^{2n-1})$ of P and Q. Not stack-clean.

`GEN ZXX_sqr_Kronecker(GEN P, long n)` return `ZX_sqr` applied to the Kronecker forms $P(X, X^{2n-1})$ of P. Not stack-clean.

7.6.10 ZXk/QXk.

`ZXk` are multivariate polynomials with `t_INT` coefficients. `QXk` are multivariate polynomials with `t_FRAC` or `t_INT` coefficients.

`GEN ZXk_divexact(GEN x, GEN y)` returns x/y assuming the quotient is a `ZXk`.

`GEN ZXk_divides(GEN x, GEN y)` returns x/y if the quotient is a `ZXk`, `NULL` otherwise.

`GEN ZXk_gcd(GEN x, GEN y)` returns a gcd of the `ZXk` x and y.

`int Rg_is_QXk(GEN z)` return 1 if `z` is a `QXk`, 0 otherwise.

`GEN QXk_gcd(GEN x, GEN y)` returns a gcd of the `QXk` x and y.

7.6.11 QX.

`void RgX_check_QX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `QX` (s should point to the name of the caller).

`GEN QX_mul(GEN x, GEN y)`

`GEN QX_sqr(GEN x)`

`GEN QX_ZX_rem(GEN x, GEN y)` y is assumed to be monic.

`GEN QX_gcd(GEN x, GEN y)` returns a gcd of the `QX` x and y.

`GEN QX_disc(GEN T)` returns the discriminant of the `QX T`.

`GEN QX_factor(GEN T)` as `ZX_factor`.

`GEN QX_resultant(GEN A, GEN B)` returns the resultant of the `QX A` and `B`.

`GEN QX_complex_roots(GEN T, long prec)` returns the complex roots of the `QX` T at accuracy `prec`, where real roots are returned as `t_REAL`s. More efficient when T is irreducible and primitive. Special case of `cleanroots`.

7.6.12 QXQ.

`GEN QXQ_norm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, using a modular algorithm. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)`, which of course does not change the norm.

If A is not a `ZX` — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call `QXQ_intnorm` instead.

`GEN QXQ_intnorm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, *assuming* that the result is an integer, which is for instance the case is $A \bmod B$ is an algebraic integer, in particular if A is a `ZX`. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)` (which of course does not change the norm).

If the result is not known to be an integer, you must use `QXQ_norm` instead, which is slower.

`GEN QXQ_mul(GEN A, GEN B, GEN T)` returns the product of A and B modulo T where both A and B are a `QX` and T is a monic `ZX`.

`GEN QXQ_sqr(GEN A, GEN T)` returns the square of A modulo T where A is a `QX` and T is a monic `ZX`.

`GEN QXQ_inv(GEN A, GEN B)` returns the inverse of A modulo B where A is a `QX` and B is a `ZX`. Should you need this for a `QX` B, just use

```
QXQ_inv(A, Q_primpart(B));
```

But in all cases where modular arithmetic modulo B is desired, it is much more efficient to replace B by `Q_primpart`(B) once and for all.

`GEN QXQ_div(GEN A, GEN B, GEN T)` returns A/B modulo T where A and B are `QX` and T is a `ZX`. Use this function when the result is expected to be of the same size as B^{-1} mod T or smaller. Otherwise, it will be faster to use `QXQ_mul(A,QXQ_inv(B,T),T)`.

`GEN QXQ_charpoly(GEN A, GEN T, long v)` where `A` is a `QX` and `T` is a `ZX`, returns the characteristic polynomial of `Mod(A, T)`. If the result is known to be a `ZX`, then calling `ZXQ_charpoly` will be faster.

`GEN QXQ_powers(GEN x, long n, GEN T)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^n]$ as `RgXQ_powers` would, but in a more efficient way when x has a huge integer denominator (we start by removing that denominator). Assume that x is a `QX` and T is a `ZX`. Meant to precompute powers of algebraic integers in $\mathbf{Q}[t]/(T)$.

`GEN QXQ_reverse(GEN f, GEN T)` as `RgXQ_reverse`, assuming f is a `QX` and T is a `ZX`.

`GEN QX_ZXQV_eval(GEN f, GEN nV, GEN dV)` as `RgX_RgXQV_eval`, except that f is assumed to be a `QX`, V is given implicitly by a numerator `nV` (`ZV`) and denominator `dV` (a positive `t_INT` or `NULL` for trivial denominator). Not stack-clean, but suitable for `gc_upto`.

`GEN QXV_QXQ_eval(GEN v, GEN a, GEN T)` v is a vector of `QX`s (possibly scalars, i.e. rational numbers, for convenience), a and T both `QX`. Return the vector of evaluations at a modulo T. Not stack-clean.

`GEN QXY_QXQ_evalx(GEN P, GEN a, GEN T)` $P(X, Y)$ is a `t_POL` with `QX` coefficients (possibly scalars, i.e. rational numbers, for convenience) , a and T both `QX`. Return the `QX` $P(a \bmod T, Y)$. Not stack-clean.

7.6.13 QXQX.

`GEN QXQX_mul(GEN x, GEN y, GEN T)` where T is a monic `ZX`.

`GEN QXQX_QXQ_mul(GEN x, GEN y, GEN T)` where T is a monic `ZX`.

`GEN QXQX_sqr(GEN x, GEN T)` where T is a monic `ZX`

`GEN QXQX_powers(GEN x, long n, GEN T)` where T is a monic `ZX`

`GEN nfgcd(GEN P, GEN Q, GEN T, GEN den)` given P and Q in $\mathbf{Z}[X, Y]$, T monic irreducible in $\mathbf{Z}[Y]$, returns the primitive d in $\mathbf{Z}[X, Y]$ which is a gcd of P, Q in $K[X]$, where K is the number field $\mathbf{Q}[Y]/(T)$. If not `NULL`, `den` is a multiple of the integral denominator of the (monic) gcd of P, Q in $K[X]$.

`GEN nfgcd_all(GEN P, GEN Q, GEN T, GEN den, GEN *Pnew)` as `nfgcd`. If `Pnew` is not `NULL`, set `*Pnew` to a nonzero integer multiple of P/d. If P and Q are both monic, then d is monic and `*Pnew` is exactly P/d. Not stack-clean if the gcd is 1 (in that case `*Pnew` is set to P).

`GEN QXQX_gcd(GEN x, GEN y, GEN T)` returns the gcd of x and y, assuming that x and y are `QXQX`s and that T is a monic `ZX`.

`GEN QXQX_homogenous_evalpow(GEN P, GEN a, GEN B, GEN T)` Evaluate the homogenous polynomial associated to the univariate polynomial P on (a, b) where B is the vector of powers of b with exponents 0 to the degree of P (`QXQ_powers(b,degpol(P),T)`).

7.6.14 QXQM.

`QXQM` are matrices of `QXQ`. All entries must be `t_INT`, `t_FRAC` or polynomials of degree strictly less than the degree of T, which must be a monic `ZX`.

`GEN QXQM_mul(GEN x, GEN y, GEN T)` returns $x * y \bmod T$.

`GEN QXQM_sqr(GEN x, GEN T)` returns $x^2 \bmod T$.

7.6.15 zx.

`GEN zero_zx(long sv)` returns a zero `zx` in variable v.

`GEN polx_zx(long sv)` returns the variable v as degree 1 `Flx`.

`GEN zx_renormalize(GEN x, long l)`, as `Flx_renormalize`, where `l` = `lg(x)`, in place.

`GEN zx_shift(GEN T, long n)` return `T` multiplied by `x`^{n}, assuming $n \geq 0$.

`long zx_lval(GEN f, long p)` return the valuation of f at p.

`GEN zx_z_divexact(GEN x, long y)` return `x/y` assuming all divisions are exact.

7.6.16 RgX.

7.6.16.1 Tests.

`long RgX_degree(GEN x, long v)` x being a `t_POL` and $v \geq 0$, returns the degree in v of x. Error if x is not a polynomial in v.

`int RgX_isscalar(GEN x)` return 1 if x all the coefficients of x of degree > 0 are 0 (as per `gequal0`).

`int RgX_is_rational(GEN P)` return 1 if the `RgX` P has only rational coefficients (`t_INT` and `t_FRAC`), and 0 otherwise.

`int RgX_is_QX(GEN P)` return 1 if the `RgX` P has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int RgX_is_ZX(GEN P)` return 1 if the `RgX` P has only `t_INT` coefficients, and 0 otherwise.

`int RgX_is_ZXX(GEN P, long *v)` return 1 if the `RgX` P is a bivariate polynomial with `t_INT` coeffcients. If `*v`≥ 0, also check that the secondary variable is equal to v, otherwise set `*v` to the secondary variable, if any.

`int RgX_is_monomial(GEN x)` returns 1 (true) if `x` is a nonzero monomial in its main variable, 0 otherwise.

`long RgX_equal(GEN x, GEN y)` returns 1 if the `t_POL`s x and y have the same `degpol` and their coefficients are equal (as per `gequal`). Variable numbers are not checked. Note that this is more stringent than `gequal(x,y)`, which only checks whether $x - y$ satisfies `gequal0`; in particular, they may have different apparent degrees provided the extra leading terms are 0.

`long RgX_equal_var(GEN x, GEN y)` returns 1 if x and y have the same variable number and `RgX_equal(x,y)` is 1.

7.6.16.2 Coefficients, blocks.

`GEN RgX_coeff(GEN P, long n)` return the coefficient of x^n in P, defined as `gen_0` if $n < 0$ or $n >$ `degpol`(P). Shallow function.

`int RgX_blocks(GEN P, long n, long m)` writes $P(X) = a_0(X) + X^n * a_1(X) * X^n + \ldots + X^{n*(m-1)}a_{m-1}(X)$, where the a_i are polynomial of degree at most $n-1$ (except possibly for the last one) and returns $[a_0(X), a_1(X), \ldots, a_{m-1}(X)]$. Shallow function.

`void RgX_even_odd(GEN p, GEN *pe, GEN *po)` write $p(X) = E(X^2) + XO(X^2)$ and set `*pe = E`, `*po = O`. Shallow function.

`GEN RgX_splitting(GEN P, long k)` write $P(X) = a_0(X^k) + Xa_1(X^k) + \ldots + X^{k-1}a_{k-1}(X^k)$ and return $[a_0(X), a_1(X), \ldots, a_{k-1}(X)]$. Shallow function.

`GEN RgX_copy(GEN x)` returns (a deep copy of) `x`.

`GEN RgX_renormalize(GEN x)` remove leading terms in `x` which are equal to (necessarily inexact) zeros.

`GEN RgX_renormalize_lg(GEN x, long lx)` as `setlg(x, lx)` followed by `RgX_renormalize(x)`. Assumes that $\texttt{lx} \leq \texttt{lg(x)}$.

`GEN RgX_recip(GEN P)` returns the reverse of the polynomial P, i.e. $X^{\deg P} P(1/X)$.

`GEN RgX_recip_shallow(GEN P)` shallow function of `RgX_recip`.

`GEN RgX_recip_i(GEN P)` shallow function of `RgX_recip`, where we further assume that $P(0) \neq 0$, so that the degree of the output is the degree of P.

`long rfracrecip(GEN *a, GEN *b)` let `*a` and `*b` be such that their quotient F is a `t_RFRAC` in variable X. Write $F(1/X) = X^v A/B$ where A and B are coprime to X and v in $\mathbf{Z}$. Set `*a` to A, `*b` to B and return v.

`GEN RgX_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q. Shallow function, may return P itself.

`long RgX_deflate_order(GEN P)` given a nonconstant polynomial P, returns the largest exponent d such that P is of the form $P(x^d)$ (use `gequal0` to check whether coefficients are 0).

`long RgX_deflate_max(GEN P, long *d)` given a nonconstant polynomial P, sets `d` to `RgX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function, may return P itself.

`long rfrac_deflate_order(GEN F)` as `RgX_deflate_order` where F is a nonconstant `t_RFRAC`.

`long rfrac_deflate_max(GEN F, long *d)` as `RgX_deflate_max` where F is a nonconstant `t_RFRAC`.

`GEN rfrac_deflate(GEN F, long m)` as `RgX_deflate` where F is a `t_RFRAC`.

`GEN RgX_inflate(GEN P, long d)` return $P(X^d)$. Shallow function.

`GEN RgX_rescale_to_int(GEN x)` given a polynomial x with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZX` wich is very close to Dx for some well-chosen integer D. More precisely, if the input is exact, D is the denominator of x; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN RgX_Graeffe(GEN P, long d)` as the GP function polgraeffe.

`GEN RgX_homogenize(GEN P, long v)` Return the homogenous polynomial associated to P in the secondary variable v, that is $y^d * P(x/y)$ where d is the degree of P, x is the variable of P, and y is the variable with number v.

`GEN RgX_homogenize_deg(GEN P, long d, long v)` Return the homogenous polynomial of degree d associated to P in the secondary variable v, that is $y^d * P(x/y)$ x is the variable of P, and y is the variable with number v.

`GEN RgXX_to_Kronecker(GEN P, long n)` Assuming $P(X, Y)$ is a bivariate polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P. Shallow function. The polynomial P is allowed to be constant in X, whether it has been simplified or not (whether X^0 appears or not in terms), but one should prefer `RgX_inflate` in this case.

On the other hand, use `RgXX_to_Kronecker_var` instead if P involves 3 variables or more: in this case, there is an ambiguity as to what X denotes and the function does not inspect all

coefficients to determine it. (Note that an inspection would only work if we could guarantee that the intended X variable appears in at least one of the terms, which is brittle and breaks if `simplify` makes it disappear. Don't take the risk.)

`GEN RgXX_to_Kronecker_var(GEN P, long n, long vx)` Safe variant of `RgXX_to_Kronecker` where `vx` is the variable number attached to X. It is assumed that its priority is less than the priority of Y, and that all other variables involved in P have lower priority than these two. Shallow function.

`GEN RgXX_to_Kronecker_spec(GEN Q, long lQ, long n)` return `RgXX_to_Kronecker`(P, n), where P is the polynomial $\sum_{i=0}^{\mathtt{lQ}-1} Q[i]x^i$. To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

7.6.16.3 Shifts, valuations.

`GEN RgX_shift(GEN x, long n)` returns $\mathtt{x} * t^n$ if $n \geq 0$, and $\mathtt{x} \backslash t^{-n}$ otherwise.

`GEN RgX_shift_shallow(GEN x, long n)` as `RgX_shift`, but shallow (coefficients are not copied).

`GEN RgX_rotate_shallow(GEN P, long k, long p)` returns $\mathtt{P} * X^k \pmod{X^p - 1}$, assuming the degree of P is strictly less than p, and $k \geq 0$.

`void RgX_shift_inplace_init(long v)` $v \geq 0$, prepare for a later call to `RgX_shift_inplace`. Reserves v words on the stack.

`GEN RgX_shift_inplace(GEN x, long v)` $v \geq 0$, assume that `RgX_shift_inplace_init`(v) has been called (reserving v words on the stack), immediately followed by a `t_POL` x. Return `RgX_shift`(x, v) by shifting x in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gc_upto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

`long RgX_valrem(GEN P, GEN *pz)` returns the valuation v of the `t_POL P` with respect to its main variable X. Check whether coefficients are 0 using `isexactzero`. Set `*pz` to `RgX_shift_shallow`$(P, -v)$.

`long RgX_val(GEN P)` returns the valuation v of the `t_POL P` with respect to its main variable X. Check whether coefficients are 0 using `isexactzero`.

`long RgX_valrem_inexact(GEN P, GEN *z)` as `RgX_valrem`, using `gequal0` instead of `isexactzero`.

`long RgXV_maxdegree(GEN V)` returns the maximum of the degrees of the components of the vector of `t_POL`s V.

7.6.16.4 Basic arithmetic.

`GEN RgX_add(GEN x, GEN y)` adds `x` and `y`.

`GEN RgX_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN RgX_neg(GEN x)` returns $-$`x`.

`GEN RgX_Rg_add(GEN y, GEN x)` returns $x + y$.

`GEN RgX_Rg_add_shallow(GEN y, GEN x)` returns $x + y$; shallow function.

`GEN Rg_RgX_sub(GEN x, GEN y)`

`GEN RgX_Rg_sub(GEN y, GEN x)` returns $x - y$

`GEN RgX_Rg_mul(GEN y, GEN x)` multiplies the `RgX` `y` by the scalar `x`.

`GEN RgX_muls(GEN y, long s)` multiplies the `RgX` `y` by the `long` `s`.

`GEN RgX_mul2n(GEN y, long n)` multiplies the `RgX` `y` by 2^n.

`GEN RgX_Rg_div(GEN y, GEN x)` divides the `RgX` `y` by the scalar `x`.

`GEN RgX_divs(GEN y, long s)` divides the `RgX` `y` by the `long` `s`.

`GEN RgX_Rg_divexact(GEN x, GEN y)` exact division of the `RgX` `y` by the scalar `x`.

`GEN RgX_Rg_eval_bk(GEN f, GEN x)` returns `f`(`x`) using Brent and Kung algorithm. (Use `poleval` for Horner algorithm.)

`GEN RgX_RgV_eval(GEN f, GEN V)` as `RgX_Rg_eval_bk(f, x)`, assuming V was output by `gpowers(x, n)` for some $n \geq 1$.

`GEN RgXV_RgV_eval(GEN f, GEN V)` apply `RgX_RgV_eval_bk(, V)` to all the components of the vector f.

`GEN RgX_normalize(GEN x)` divides x by its leading coefficient. If the latter is 1, x itself is returned, not a copy. Leading coefficients equal to 0 are stripped, e.g.

```
0.*t^3 + Mod(0,3)*t^2 + 2*t
```

is normalized to t.

`GEN RgX_mul(GEN x, GEN y)` multiplies the two `t_POL` (in the same variable) `x` and `y`. Detect the coefficient ring and use an appropriate algorithm.

`GEN RgX_mul_i(GEN x, GEN y)` multiplies the two `t_POL` (in the same variable) `x` and `y`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

`GEN RgX_mul_normalized(GEN A, long a, GEN B, long b)` returns $(X^a + A)(X^b + B) - X^(a+b)$, where we assume that $\deg A < a$ and $\deg B < b$ are polynomials in the same variable X.

`GEN RgX_sqr(GEN x)` squares the `t_POL` `x`. Detect the coefficient ring and use an appropriate algorithm.

`GEN RgX_sqr_i(GEN x)` squares the `t_POL` `x`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

`GEN RgXV_prod(GEN V)`, `V` being a vector of `RgX`, returns their product.

`GEN RgX_divrem(GEN x, GEN y, GEN *r)` by default, returns the Euclidean quotient and store the remainder in r. Three special values of r change that behavior • `NULL`: do not store the remainder, used to implement `RgX_div`,

• `ONLY_REM`: return the remainder, used to implement `RgX_rem`,

• `ONLY_DIVIDES`: return the quotient if the division is exact, and `NULL` otherwise.

In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`.

`GEN RgX_div(GEN x, GEN y)`

`GEN RgX_div_by_X_x(GEN A, GEN a, GEN *r)` returns the quotient of the `RgX A` by $(X - \mathtt{a})$, and sets `r` to the remainder `A`(`a`).

`GEN RgX_rem(GEN x, GEN y)`

`GEN RgX_pseudodivrem(GEN x, GEN y, GEN *ptr)` compute a pseudo-quotient q and pseudo-remainder r such that $\mathtt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$. Return q and set `*ptr` to r.

`GEN RgX_pseudorem(GEN x, GEN y)` return the remainder in the pseudo-division of x by y.

`GEN RgXQX_pseudorem(GEN x, GEN y, GEN T)` return the remainder in the pseudo-division of x by y over $R[X]/(T)$.

`int ZXQX_dvd(GEN x, GEN y, GEN T)` let T be a monic irreducible `ZX`, let x, y be `t_POL` whose coefficients are either `t_INT`s or `ZX` in the same variable as T. Assume further that the leading coefficient of y is an integer. Return 1 if $y|x$ in $(\mathbf{Z}[Y]/(T))[X]$, and 0 otherwise.

`GEN RgXQX_pseudodivrem(GEN x, GEN y, GEN T, GEN *ptr)` compute a pseudo-quotient q and pseudo-remainder r such that $\mathtt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ in $R[X]/(T)$. Return q and set `*ptr` to r.

`GEN RgX_mulXn(GEN a, long n)` returns $a * X^n$. This may be a `t_FRAC` if $n < 0$ and the valuation of a is not large enough.

`GEN RgX_addmulXn(GEN a, GEN b, long n)` returns $a + b * X^n$, assuming that $n > 0$.

`GEN RgX_addmulXn_shallow(GEN a, GEN b, long n)` shallow variant of `RgX_addmulXn`.

`GEN RgX_digits(GEN x, GEN B)` returns a vector of `RgX` $[c_0, \ldots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

7.6.16.5 Internal routines working on coefficient arrays.

These routines operate on coefficient blocks which are invalid `GEN`s A `GEN` argument a or b in routines below is actually a coefficient arrays representing the polynomials $\sum_{i=0}^{\mathtt{na}-1} a[i]X^i$ and $\sum_{i=0}^{\mathtt{nb}-1} b[i]X^i$. Note that $a[0]$ and $b[0]$ contain coefficients and not the mandatory `GEN` codeword. This allows to implement divide-and-conquer methods directly, without needing to allocate wrappers around coefficient blocks.

`GEN RgX_mulspec(GEN a, GEN b, long na, long nb)`. Internal routine: given two coefficient arrays representing polynomials, return their product (as a true `GEN`) in variable 0.

`GEN RgX_sqrspec(GEN a, long na)`. Internal routine: given a coefficient array representing a polynomial r eturn its square (as a true `GEN`) in variable 0.

`GEN RgX_addspec(GEN x, GEN y, long nx, long ny)` given two coefficient arrays representing polynomials, return their sum (as a true `GEN`) in variable 0.

`GEN RgX_addspec_shallow(GEN x, GEN y, long nx, long ny)` shallow variant of `RgX_addspec`.

7.6.16.6 GCD, Resultant.

`GEN RgX_gcd(GEN x, GEN y)` returns the GCD of `x` and `y`, assumed to be `t_POL`s in the same variable.

`GEN RgX_gcd_simple(GEN x, GEN y)` as `RgX_gcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_gcd`.

`GEN RgX_extgcd(GEN x, GEN y, GEN *u, GEN *v)` returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$, and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{u} * \mathtt{x} + *\mathtt{v} * \mathtt{y} = d$. Uses a generic subresultant algorithm.

`GEN RgX_extgcd_simple(GEN x, GEN y, GEN *u, GEN *v)` as `RgX_extgcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_extgcd`.

`GEN RgX_halfgcd(GEN x, GEN y)` assuming `x` and `y` are `t_POL`s in the same variable, returns a 2×2 `t_MAT` M with `t_POL` entries, such that $M * [x, y]^{\mathtt{T}} == [a, b]^{\mathtt{T}}$ such that $\deg a \geq \lceil \max(\deg x, \deg y))/2 \rceil > \deg b$.

`GEN RgX_halfgcd_all(GEN x, GEN y, GEN *pt_a, GEN *pt_b)` as `RgX_halfgcd`, in addition, if `pt_a` (resp. `pt_b`) is not `NULL`, `*pt_a` (resp. `*pt_b`) is set to `a` (resp. `b`).

`GEN RgX_chinese_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz)` returns an `RgX`, congruent to `x` mod `Tx` and to `y` mod `Ty`. Assumes `Tx` and `Ty` are coprime, and `Tz = Tx * Ty` or `NULL` (in which case it is computed within).

`GEN RgX_disc(GEN x)` returns the discriminant of the `t_POL` `x` with respect to its main variable.

`GEN RgX_resultant_all(GEN x, GEN y, GEN *sol)` returns `resultant(x,y)`. If `sol` is not `NULL`, sets it to the last nonconstant remainder in the polynomial remainder sequence if it exists and to `gen_0` otherwise (e.g. one polynomial has degree 0).

7.6.16.7 Other operations.

`GEN RgX_gtofp(GEN x, GEN prec)` returns the polynomial obtained by applying

```
gtofp(gel(x,i), prec)
```

to all coefficients of x.

`GEN RgX_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgX_gtofp(x, prec) )
```

`GEN RgX_deriv(GEN x)` returns the derivative of `x` with respect to its main variable.

`GEN RgX_derivn(GEN x, long n)` returns the nth derivative of `x` with respect to its main variable.

`GEN RgX_integ(GEN x)` returns the primitive of `x` vanishing at 0, with respect to its main variable.

`GEN RgX_rescale(GEN P, GEN h)` returns $h^{\deg(P)} P(x/h)$. `P` is an `RgX` and `h` is nonzero. (Leaves small objects on the stack. Suitable but inefficient for `gc_upto`.)

`GEN RgXV_rescale(GEN v, GEN h)` apply `RgX_unscale` to a vector of `RgX`.

`GEN RgX_unscale(GEN P, GEN h)` returns $P(hx)$. (Leaves small objects on the stack. Suitable but inefficient for `gc_upto`.)

`GEN RgXV_unscale(GEN v, GEN h)` apply `RgX_unscale` to a vector of `RgX`.

`GEN RgX_Rg_translate(GEN P, GEN c)` assume c is a scalar or a polynomials whose main variable has lower priority than the main variable X of P. Returns $P(X + c)$ (optimized for $c = \pm 1$).

`GEN RgX_affine(GEN P, GEN a, GEN b)` Return $P(aX + b)$ (optimized for $b = \pm 1$). Not stack-clean.

7.6.16.8 Implicit bivariate homogeneous polynomials.

A bivariate homogeneous polynomial $P(X,Y)$ of (homogeneous) degree d is represented by giving $P(X,1)$ and d.

`GEN RgX_homogenous_derivn(GEN P, long n, long d)` returns the bivariate homogeneous polynomial of homogeneous degree $d-n$ corresponding to the n-th derivative with respect to the implicit variables of the homogeneous polynomial P of degree d.

`GEN RgX_homogenous_eval(GEN P, GEN a, GEN b, long d)` Evaluate the homogeneous polynomial of degree d associated to the univariate polynomial P on (a,b).

`GEN RgX_homogenous_evalpow(GEN P, GEN a, GEN B, long d)` Evaluate the homogeneous polynomial of degree d associated to the univariate polynomial P on (a,b) where B is the vector of powers of b with exponents 0 to d (`gpowers(b,d)`).

7.6.16.9 Function related to modular forms.

`GEN RgX_act_Gl2Q(GEN g, long k)` let R be a commutative ring and $g=[a,b;c,d]$ be in $\mathrm{GL}_2(\mathbf{Q})$, g acts (on the left) on homogeneous polynomials of degree $k-2$ in $V:=R[X,Y]_{k-2}$ via

$$g\cdot P := P(dX-cY,-bX+aY) = (\det g)^{k-2}P((X,Y)\cdot g^{-1}).$$

This function returns the matrix in $M_{k-1}(R)$ of $P\mapsto g\cdot P$ in the basis $(X^{k-2},\dots,Y^{k-2})$ of V.

`GEN RgX_act_ZGl2Q(GEN z, long k)` let $G:=\mathrm{GL}_2(\mathbf{Q})$, acting on $R[X,Y]_{k-2}$ and $z\in\mathbf{Z}[G]$. Return the matrix giving $P\mapsto z\cdot P$ in the basis $(X^{k-2},\dots,Y^{k-2})$.

7.6.17 RgXn.

`GEN RgXn_red_shallow(GEN x, long n)` return $\mathtt{x}\%t^n$, where $n\geq 0$. Shallow function.

`GEN RgXn_recip_shallow(GEN P)` returns $X^nP(1/X)$. Shallow function.

`GEN RgXn_mul(GEN a, GEN b, long n)` returns ab modulo X^n, where a,b are two `t_POL` in the same variable X and $n\geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgXn_sqr(GEN a, long n)` returns a^2 modulo X^n, where a is a `t_POL` in the variable X and $n\geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgX_mulhigh_i(GEN f, GEN g, long n)` return the Euclidean quotient of $f(x)*g(x)$ by x^n (high product). Uses `RgXn_mul` applied to the reciprocal polynomials of f and g. Shallow function.

`GEN RgX_sqrhigh_i(GEN f, long n)` return the Euclidean quotient of $f(x)^2$ by x^n (high product). Uses `RgXn_sqr` applied to the reciprocal polynomial of f. Shallow function.

`GEN RgXn_inv(GEN a, long n)` returns a^{-1} modulo X^n, where a is a `t_POL` in the variable X and $n\geq 0$. Uses Newton-Raphson algorithm.

`GEN RgXn_inv_i(GEN a, long n)` as `RgXn_inv` without final garbage collection (suitable for `gc_upto`).

`GEN RgXn_div(GEN a, GEN b, long n)` returns a/b modulo X^n, where a and b are `t_POL`s in the variable X and $n\geq 0$. Uses Newton-Raphson/Karp-Markstein algorithm.

`GEN RgXn_div_i(GEN a, GEN b, long n)` as `RgXn_div` without final garbage collection (suitable for `gc_upto`).

GEN RgXn_powers(GEN x, long m, long n) returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{m}}]$ modulo X^n as a t_VEC of RgXns.

GEN RgXn_powu(GEN x, ulong m, long n) returns x^m modulo X^n.

GEN RgXn_powu_i(GEN x, ulong m, long n) as RgXn_powu, not stack-clean.

GEN RgXn_sqrt(GEN a, long n) returns $a^{1/2}$ modulo X^n, where a is a t_POL in the variable X and $n \geq 0$. Assume that $a = 1 \bmod X$. Uses Newton algorithm.

GEN RgXn_exp(GEN a, long n) returns $exp(a)$ modulo X^n, assuming $a = 0 \bmod X$.

GEN RgXn_expint(GEN f, long n) return $\exp(F)$ where F is the primitive of f that vanishes at 0.

GEN RgXn_eval(GEN Q, GEN x, long n) special case of RgX_RgXQ_eval, when the modulus is a monomial: returns Q(x) modulo t^n, where $x \in R[t]$.

GEN RgX_RgXn_eval(GEN f, GEN x, long n) returns f(x) modulo X^n.

GEN RgX_RgXnV_eval(GEN f, GEN V, long n) as RgX_RgXn_eval(f, x, n), assuming V was output by RgXn_powers(x, m, n) for some $m \geq 1$.

GEN RgXn_reverse(GEN f, long n) assuming that $f = ax \bmod x^2$ with a invertible, returns a t_POL g of degree $< n$ such that $(g \circ f)(x) = x$ modulo x^n.

7.6.18 RgXnV.

GEN RgXnV_red_shallow(GEN x, long n) apply RgXn_red_shallow to all the components of the vector x.

7.6.19 RgXQ.

GEN RgXQ_mul(GEN y, GEN x, GEN T) computes $xy \bmod T$

GEN RgXQ_sqr(GEN x, GEN T) computes $x^2 \bmod T$

GEN RgXQ_inv(GEN x, GEN T) return the inverse of $x \bmod T$.

GEN RgXQ_pow(GEN x, GEN n, GEN T) computes $x^n \bmod T$

GEN RgXQ_powu(GEN x, ulong n, GEN T) computes $x^n \bmod T$, n being an ulong.

GEN RgXQ_powers(GEN x, long n, GEN T) returns $[\mathtt{x}^0, \ldots, \mathtt{x}^{\mathtt{n}}]$ as a t_VEC of RgXQs.

GEN RgXQ_matrix_pow(GEN y, long n, long m, GEN P) returns RgXQ_powers(y,m-1,P), as a matrix of dimension $n \geq \deg P$.

GEN RgXQ_norm(GEN x, GEN T) returns the norm of Mod(x, T).

GEN RgXQ_trace(GEN x, GEN T) returns the trace of Mod(x, T).

GEN RgXQ_charpoly(GEN x, GEN T, long v) returns the characteristic polynomial of Mod(x, T), in variable v.

GEN RgXQ_minpoly(GEN x, GEN T, long v) returns the minimal polynomial of Mod(x, T), in variable v.

GEN RgX_RgXQ_eval(GEN f, GEN x, GEN T) returns f(x) modulo T.

GEN RgX_RgXQV_eval(GEN f, GEN V, GEN T) as RgX_RgXQ_eval(f, x, T), assuming V was output by RgXQ_powers(x, n, T) for some $n \geq 1$.

`int RgXQ_ratlift(GEN x, GEN T, long amax, long bmax, GEN *P, GEN *Q)` Assuming that `amax` + `bmax` $< \deg T$, attempts to recognize x as a rational function a/b, i.e. to find `t_POL`s P and Q such that

- $P \equiv Qx$ modulo T,
- $\deg P \leq$ `amax`, $\deg Q \leq$ `bmax`,
- $\gcd(T, P) = \gcd(P, Q)$.

If unsuccessful, the routine returns 0 and leaves P, Q unchanged; otherwise it returns 1 and sets P and Q.

`GEN RgXQ_reverse(GEN f, GEN T)` returns a `t_POL` g of degree $< n = \deg T$ such that $T(x)$ divides $(g \circ f)(x) - x$, by solving a linear system. Low-level function underlying `modreverse`: it returns a lift of [modreverse(f,T)]; faster than the high-level function since it needs not compute the characteristic polynomial of f mod T (often already known in applications). In the trivial case where $n \leq 1$, returns a scalar, not a constant `t_POL`.

7.6.20 `RgXQV, RgXQC`.

`GEN RgXQC_red(GEN z, GEN T)` `z` a vector whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_COL`.

`GEN RgXQV_red(GEN z, GEN T)` `z` a vector whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_VEC`.

`GEN RgXQV_RgXQ_mul(GEN z, GEN x, GEN T)` `z` multiplies the `RgXQV` `z` by the scalar (`RgXQ`) `x`.

`GEN RgXQV_factorback(GEN L, GEN e, GEN T)` returns $\prod_i L_i^{e_i}$ mod T where L is a vector of `RgXQ`s and e a vector of `t_INT`s.

7.6.21 `RgXQM`.

`GEN RgXQM_red(GEN z, GEN T)` `z` a matrix whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise).

`GEN RgXQM_mul(GEN x, GEN y, GEN T)`

7.6.22 `RgXQX`.

`GEN RgXQX_red(GEN z, GEN T)` `z` a `t_POL` whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise).

`GEN RgXQX_mul(GEN x, GEN y, GEN T)`

`GEN RgXQX_RgXQ_mul(GEN x, GEN y, GEN T)` multiplies the `RgXQX` `y` by the scalar (`RgXQ`) `x`.

`GEN RgXQX_sqr(GEN x, GEN T)`

`GEN RgXQX_powers(GEN x, long n, GEN T)`

`GEN RgXQX_divrem(GEN x, GEN y, GEN T, GEN *pr)`

`GEN RgXQX_div(GEN x, GEN y, GEN T)`

`GEN RgXQX_rem(GEN x, GEN y, GEN T)`

`GEN RgXQX_RgXQ_translate(GEN P, GEN c, GEN T)` assume the main variable X of P has higher priority than the main variable Y of T and c. Return a lift of $P(X + \mathrm{Mod}(c(Y), T(Y)))$.

`GEN Kronecker_to_mod(GEN z, GEN T)` $z \in R[X]$ represents an element $P(X,Y)$ in $R[X,Y]$ mod $T(Y)$ in Kronecker form, i.e. $z = P(X, X^{2*n-1})$

Let R be some commutative ring, $n = \deg T$ and let $P(X,Y) \in R[X,Y]$ lift a polynomial in $K[Y]$, where $K := R[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns the image of $P(X,t)$ in $K[t]$, with `t_POLMOD` coefficients. Not stack-clean. Note that t need not be the same variable as Y!

Chapter 8:
Black box algebraic structures

The generic routines like `gmul` or `gadd` allow handling objects belonging to a fixed list of basic types, with some natural polymorphism (you can mix rational numbers and polynomials, etc.), at the expense of efficiency and sometimes of clarity when the recursive structure becomes complicated, e.g. a few levels of `t_POLMOD`s attached to different polynomials and variable numbers for quotient structures. This is the only possibility in GP.

On the other hand, the Level 2 Kernel allows dedicated routines to handle efficiently objects of a very specific type, e.g. polynomials with coefficients in the same finite field. This is more efficient, but imvolves a lot of code duplication since polymorphism is no longer possible.

A third and final option, still restricted to library programming, is to define an arbitrary algebraic structure (currently groups, fields, rings, algebras and $\mathbf{Z}_p$-modules) by providing suitable methods, then using generic algorithms. For instance naive Gaussian pivoting applies over all base fields and need only be implemented once. The difference with the first solution is that we no longer depend on the way functions like `gmul` or `gadd` will guess what the user is trying to do. We can then implement independently various groups / fields / algebras in a clean way.

8.1 Black box groups.

A black box group is defined by a `bb_group` struct, describing methods available to handle group elements:

```
struct bb_group
{
  GEN (*mul)(void*, GEN, GEN);
  GEN (*pow)(void*, GEN, GEN);
  GEN (*rand)(void*);
  ulong (*hash)(GEN);
  int (*equal)(GEN, GEN);
  int (*equal1)(GEN);
  GEN (*easylog)(void *E, GEN, GEN, GEN);
};
```

`mul(E,x,y)` returns the product xy.

`pow(E,x,n)` returns x^n (n integer, possibly negative or zero).

`rand(E)` returns a random element in the group.

`hash(x)` returns a hash value for x (`hash_GEN` is suitable for this field).

`equal(x,y)` returns one if $x = y$ and zero otherwise.

`equal1(x)` returns one if x is the neutral element in the group, and zero otherwise.

`easylog(E,a,g,o)` (optional) returns either NULL or the discrete logarithm n such that $g^n = a$, the element g being of order o. This provides a short-cut in situation where a better algorithm than the generic one is known.

A group is thus described by a `struct bb_group` as above and auxiliary data typecast to `void*`. The following functions operate on black box groups:

`GEN gen_Shanks_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)` Generic baby-step/giant-step algorithm (Shanks's method). Assuming that g has order N, compute an integer k such that $g^k = x$. Return `cgetg(1, t_VEC)` if there are no solutions. This requires $O(\sqrt{N})$ group operations and uses an auxiliary table containing $O(\sqrt{N})$ group elements.

The above is useful for a one-shot computation. If many discrete logs are desired: `GEN gen_Shanks_init(GEN g, long n, void *E, const struct bb_group *grp)` return an auxiliary data structure T required to compute a discrete log in base g. Compute and store all powers g^i, $i < n$.

`GEN gen_Shanks(GEN T, GEN x, ulong N, void *E, const struct bb_group *grp)` Let T be computed by `gen_Shanks_init`$(g, n, \ldots)$. Return $k < nN$ such that $g^k = x$ or `NULL` if no such index exist. It uses $O(N)$ operation in the group and fast table lookups (in time $O(\log n)$). The interface is such that the function may be used when the order of the base g is unknown, and hence compute it given only an upper bound B for it: e.g. choose n, N such that $nN \geq B$ and compute the discrete log l of g^{-1} in base g, then use `gen_order` with multiple $N = l + 1$.

`GEN gen_Pollard_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)` Generic Pollard rho algorithm. Assuming that g has order N, compute an integer k such that $g^k = x$. This requires $O(\sqrt{N})$ group operations in average and $O(1)$ storage. Will enter an infinite loop if there are no solutions.

`GEN gen_plog(GEN x, GEN g, GEN N, void *E, const struct bb_group)` Assuming that g has prime order N, compute an integer k such that $g^k = x$, using either `gen_Shanks_log` or `gen_Pollard_log`. Return `cgetg(1, t_VEC)` if there are no solutions.

`GEN gen_Shanks_sqrtn(GEN a, GEN n, GEN N, GEN *zetan, void *E, const struct bb_group *grp)` returns one solution of $x^n = a$ in a black box cyclic group of order N. Return `NULL` if no solution exists. If `zetan` is not `NULL` it is set to an element of exact order n. This function uses `gen_plog` for all prime divisors of $\gcd(n, N)$.

`GEN gen_PH_log(GEN a, GEN g, GEN N, void *E, const struct bb_group *grp)` returns an integer k such that $g^k = x$, assuming that the order of g divides N, using Pohlig-Hellman algorithm. Return `cgetg(1, t_VEC)` if there are no solutions. This calls `gen_plog` repeatedly for all prime divisors p of N.

In the following functions the integer parameter `ord` can be given in all the formats recognized for the argument of arithmetic functions, i.e. either as a positive `t_INT` N, or as its factorization matrix faN, or (preferred) as a pair $[N, faN]$.

`GEN gen_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` computes the order of x; `ord` is a multiple of the order, for instance the group order.

`GEN gen_factored_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` returns a pair $[o, F]$, where o is the order of x and F is the factorization of o; `ord` is as in `gen_order`.

`GEN gen_gener(GEN ord, void *E, const struct bb_group *grp)` returns a random generator of the group, assuming it is of order exactly `ord`.

`GEN get_arith_Z(GEN ord)` given `ord` as above in one of the formats recognized for arithmetic functions, i.e. a positive `t_INT` N, its factorization faN, or the pair $[N, faN]$, return N.

`GEN get_arith_ZZM(GEN ord)` given `ord` as above, return the pair $[N, faN]$. This may require factoring N.

`GEN gen_select_order(GEN v, void *E, const struct bb_group *grp)` Let v be a vector of possible orders for the group; try to find the true order by checking orders of random points. This will not terminate if there is an ambiguity.

8.1.1 Black box groups with pairing.

These functions handle groups of rank at most 2 equipped with a family of bilinear pairings which behave like the Weil pairing on elliptic curves over finite field. In the descriptions below, the function `pairorder(E, P, Q, m, F)` must return the order of the m-pairing of P and Q, both of order dividing m, where F is the factorization matrix of a multiple M of m. The format $[M, \texttt{factor}(M)]$ is also accepted for F, and in fact preferred.

`GEN gen_ellgroup(GEN o, GEN d, GEN *pt_m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` returns the elementary divisors $[d_1, d_2]$ of the group, assuming it is of order exactly $o > 1$, and that d_2 divides d. If $d_2 = 1$ then $[o]$ is returned, otherwise `m=*pt_m` is set to the order of the pairing required to verify a generating set which is to be used with `gen_ellgens`. For the parameter o, all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \texttt{factor}(n)]$.

`GEN gen_ellgens(GEN d1, GEN d2, GEN m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` the parameters d_1, d_2, m being as returned by `gen_ellgroup`, returns a pair of generators $[P, Q]$ such that P is of order d_1 and the m-pairing of P and Q is of order m. (Note: Q needs not be of order d_2). For the parameter d_1, all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \texttt{factor}(n)]$.

8.1.2 Functions returning black box groups.

`const struct bb_group * get_Flxq_star(void **E, GEN T, ulong p)`

`const struct bb_group * get_FpXQ_star(void **E, GEN T, GEN p)` returns a pointer to the black box group $(\mathbf{F}_p[x]/(T))^*$.

`const struct bb_group * get_FpE_group(void **pE, GEN a4, GEN a6, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p)$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p$.

`const struct bb_group * get_FpXQE_group(void **pE, GEN a4, GEN a6, GEN T, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p[X]/(T))$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$.

`const struct bb_group * get_FlxqE_group(void **pE, GEN a4, GEN a6, GEN T, ulong p)` idem for small p.

`const struct bb_group * get_F2xqE_group(void **pE, GEN a2, GEN a6, GEN T)` idem for $p = 2$.

8.2 Black box fields.

A black box field is defined by a `bb_field` struct, describing methods available to handle field elements:

```
struct bb_field
{
  GEN (*red)(void *E ,GEN);
  GEN (*add)(void *E ,GEN, GEN);
  GEN (*mul)(void *E ,GEN, GEN);
  GEN (*neg)(void *E ,GEN);
  GEN (*inv)(void *E ,GEN);
  int (*equal0)(GEN);
  GEN (*s)(void *E, long);
};
```

In contrast of black box group, elements can have non canonical forms, and only `red` is required to return a canonical form. For instance a black box implementation of finite fields, all methods except `red` may return arbitrary representatives in $\mathbf{Z}[X]$ of the correct congruence class modulo $(p, T(X))$.

`red(E,x)` returns the canonical form of x.

`add(E,x,y)` returns the sum $x + y$.

`mul(E,x,y)` returns the product xy.

`neg(E,x)` returns $-x$.

`inv(E,x)` returns the inverse of x.

`equal0(x)` x being in canonical form, returns one if $x = 0$ and zero otherwise.

`s(n)` n being a small signed integer, returns n times the unit element.

A field is thus described by a `struct bb_field` as above and auxiliary data typecast to `void*`. The following functions operate on black box fields:

`GEN gen_Gauss(GEN a, GEN b, void *E, const struct bb_field *ff)`

`GEN gen_Gauss_pivot(GEN x, long *rr, void *E, const struct bb_field *ff)`

`GEN gen_det(GEN a, void *E, const struct bb_field *ff)`

`GEN gen_ker(GEN x, long deplin, void *E, const struct bb_field *ff)`

`GEN gen_matcolinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)`

`GEN gen_matcolmul(GEN a, GEN b, void *E, const struct bb_field *ff)`

`GEN gen_matid(long n, void *E, const struct bb_field *ff)`

`GEN gen_matinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)`

`GEN gen_matmul(GEN a, GEN b, void *E, const struct bb_field *ff)`

8.2.1 Functions returning black box fields.

`const struct bb_field * get_Fp_field(void **pE, GEN p)`

`const struct bb_field * get_Fq_field(void **pE, GEN T, GEN p)`

`const struct bb_field * get_Flxq_field(void **pE, GEN T, ulong p)`

`const struct bb_field * get_F2xq_field(void **pE, GEN T)`

`const struct bb_field * get_nf_field(void **pE, GEN nf)`

8.3 Black box algebra.

A black box algebra is defined by a `bb_algebra` struct, describing methods available to handle algebra elements:

```
struct bb_algebra
{
  GEN (*red)(void *E, GEN x);
  GEN (*add)(void *E, GEN x, GEN y);
  GEN (*sub)(void *E, GEN x, GEN y);
  GEN (*mul)(void *E, GEN x, GEN y);
  GEN (*sqr)(void *E, GEN x);
  GEN (*one)(void *E);
  GEN (*zero)(void *E);
};
```

In contrast with black box groups, elements can have non canonical forms, but only `add` is allowed to return a non canonical form.

`red(E,x)` returns the canonical form of x.

`add(E,x,y)` returns the sum $x + y$.

`sub(E,x,y)` returns the difference $x - y$.

`mul(E,x,y)` returns the product xy.

`sqr(E,x)` returns the square x^2.

`one(E)` returns the unit element.

`zero(E)` returns the zero element.

An algebra is thus described by a `struct bb_algebra` as above and auxiliary data typecast to `void*`. The following functions operate on black box algebra:

`GEN gen_bkeval(GEN P, long d, GEN x, int use_sqr, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))` x being an element of the black box algebra, and P some black box polynomial of degree d over the base field, returns $P(x)$. The function `cmul(E,P,a,y)` must return the coefficient of degree a of P multiplied by y. `cmul` is allowed to return a non canonical form; it is also allowed to return `NULL` instead of an exact 0.

The flag `use_sqr` has the same meaning as for `gen_powers`. This implements an algorithm of Brent and Kung (1978).

`GEN gen_bkeval_powers(GEN P, long d, GEN V, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))` as `gen_RgX_bkeval` assuming V was output by `gen_powers`(x, l, E, ff) for some $l \geq 1$. For optimal performance, l should be computed by `brent_kung_optpow`.

`long brent_kung_optpow(long d, long n, long m)` returns the optimal parameter l for the evaluation of n/m polynomials of degree d. Fractional values can be used if the evaluations are done with different accuracies, and thus have different weights.

8.3.1 Functions returning black box algebras.

`const struct bb_algebra * get_FpX_algebra(void **E, GEN p, long v)` return the algebra of polynomials over $\mathbf{F}_p$ in variable v.

`const struct bb_algebra * get_Flx_algebra(void **E, ulong p, long v)` return the algebra of polynomials over $\mathbf{F}_p$ in variable v (for `ulong` p).

`const struct bb_algebra * get_FpXQ_algebra(void **E, GEN T, GEN p)` return the algebra $\mathbf{F}_p[X]/(T(X))$.

`const struct bb_algebra * get_Flxq_algebra(void **E, GEN T, ulong p)` return the algebra $\mathbf{F}_p[X]/(T(X))$ (for `ulong` p).

`const struct bb_algebra * get_FpXQX_algebra(void **E, GEN T, GEN p, long v)` return the algebra of polynomials over $\mathbf{F}_p[X]/(T(X))$ in variable v.

`const struct bb_algebra * get_FlxqX_algebra(void **E, GEN T, ulong p, long v)` return the algebra of polynomials over $\mathbf{F}_p[X]/(T(X))$ in variable v (for `ulong` p).

`const struct bb_algebra * get_FpXQXQ_algebra(void **E, GEN S, GEN T, GEN p)` return the algebra $\mathbf{F}_p[X,Y]/(S(X,Y), T(X))$.

`const struct bb_algebra * get_FlxqXQ_algebra(void **E, GEN S, GEN T, ulong p)` return the algebra $\mathbf{F}_p[X,Y]/(S(X,Y), T(X))$ (for `ulong` p).

`const struct bb_algebra * get_Rg_algebra(void)` return the generic algebra (`E` is not used and can be set to `NULL`).

8.4 Black box ring.

A black box ring is defined by a `bb_ring` struct, describing methods available to handle ring elements:

```
struct bb_ring
{
  GEN (*add)(void *E, GEN x, GEN y);
  GEN (*mul)(void *E, GEN x, GEN y);
  GEN (*sqr)(void *E, GEN x);
};
```

`add(E,x,y)` returns the sum $x + y$.

`mul(E,x,y)` returns the product xy.

`sqr(E,x)` returns the square x^2.

`GEN gen_fromdigits(GEN v, GEN B, void *E, struct bb_ring *r)` where B is a ring element and $v = [c_0, \ldots, c_{n-1}]$ a vector of ring elements, return $\sum_{i=0}^{n} c_i B^i$ using binary splitting.

`GEN gen_digits(GEN x, GEN B, long n, void *E, struct bb_ring *r, GEN (*div)(void *E, GEN x, GEN y, GEN *r))`

(Require the ring to be Euclidean)

`div(E,x,y,&r)` performs the Euclidean division of x by y in the ring R, returning the quotient q and setting r to the residue so that $x = qy + r$ holds. The residue must belong to a fixed set of representatives of $R/(y)$.

The argument x being a ring element, `gen_digits` returns a vector of ring elements $[c_0, \ldots, c_{n-1}]$ such that $x = \sum_{i=0}^{n} c_i B^i$. Furthermore for all $i \neq n-1$, the elements c_i belonging to the fixed set of representatives of $R/(B)$.

8.5 Black box free $\mathbf{Z}_p$-modules.

(Very experimental)

`GEN gen_ZpX_Dixon(GEN F, GEN V, GEN q, GEN p, long N, void *E, GEN lin(void *E, GEN F, GEN z, GEN q), GEN inv(void *E, GEN z))`

Let F be a `ZpXT` representing the coefficients of some abstract linear mapping f over $\mathbf{Z}_p[X]$ seen as a free $\mathbf{Z}_p$-module, let V be an element of $\mathbf{Z}_p[X]$ and let $q = p^N$. Return $y \in \mathbf{Z}_p[X]$ such that $f(y) = V \pmod{p^N}$ assuming the following holds for $n \leq N$:

- $\texttt{lin}(E, \texttt{FpX\_red}(F, p^n), z, p^n) \equiv f(z) \pmod{p^n}$
- $f(\texttt{inv}(E, z)) \equiv z \pmod{p}$

The argument F is redundant; including it allows the function to reduce it to the required p-adic precision when evaluating f modulo p^n.

`GEN gen_ZpX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))`

Let x be an element of $\mathbf{Z}_p[X]$ seen as a free $\mathbf{Z}_p$-module, and f some differentiable function over $\mathbf{Z}_p[X]$ such that $f(x) \equiv 0 \pmod{p}$. Return y such that $f(y) \equiv 0 \pmod{p^n}$, assuming the following holds for all $a, b \in \mathbf{Z}_p[X]$ and $M \leq N$:

- $v = \texttt{eval}(E, a, p^N)$ is a vector of elements of $\mathbf{Z}_p[X]$,
- $w = \texttt{invd}(E, b, v, p^M, M)$ is an element in $\mathbf{Z}_p[X]$,
- $v[1] \equiv f(a) \pmod{p^N \mathbf{Z}_p[X]}$,
- $df_a(w) \equiv b \pmod{p^M \mathbf{Z}_p[X]}$

and df_a denotes the differential of f at a. Motivation: `eval` allows to evaluate f and `invd` allows to invert its differential. Frequently, data useful to compute the differential appear as a subproduct of computing the function. The vector v allows `eval` to provide these to `invd`. The implementation of `invd` will generally involves the use of the function `gen_ZpX_Dixon`.

`GEN gen_ZpM_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))` as above, with polynomials replaced by matrices.

`GEN gen_ZpXX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))` as above, with `ZpX` replaced by `ZpXX`.

Chapter 9:
Operations on general PARI objects

9.1 Conversions.

9.1.1 Scalars.

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

`double rtodbl(GEN x)` applied to a `t_REAL` `x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double` `x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`int gisdouble(GEN x, double *dx)` if `x` is a real number (not necessarily a `t_REAL`) and if `x` can be converted to a `double`, set `*dx` to the converted value and return 1. Otherwise return 0.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT` `x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`ulong gtou(GEN x)` converts the non-negative `t_INT` `x` to an unsigned small integer if possible, otherwise raise an exception. This function is similar to `itou`, slightly slower since it checks the type of `x`.

`double dblmodulus(GEN x)` return an approximation to `|x|`.

`long gtolong(GEN x)` if `x` is an integer (not necessarily a `t_INT`), converts `x` into a `long` if possible.

`GEN fractor_lg(GEN x, long l)` applied to a `t_FRAC` `x`, converts `x` into a `t_REAL` of length `l`.

`GEN fractor(GEN x, long prec)` applied to a `t_FRAC` `x`, converts `x` into a `t_REAL` of precision `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD` `x`, converts `x` into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of `x`, to precision `l`.

`GEN upper_to_cx(GEN x, long *prec)` valid for a `t_COMPLEX` or `t_QUAD` belonging to the upper half-plane. If a `t_QUAD`, convert it to `t_COMPLEX` using accuracy `*prec`. If x is inexact, sets `*prec` to the precision of x.

`GEN cxtofp(GEN x, long prec)` converts the `t_COMPLEX` `x` to a a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`.

`GEN cxcompotor(GEN x, long prec)` converts the `t_INT`, `t_REAL` or `t_FRAC` x to a `t_REAL` of length `prec`. These are all the real types which may occur as components of a `t_COMPLEX`; special case of `gtofp` (introduced so that the latter is not recursive and can thus be inlined).

`GEN cxtoreal(GEN x)` converts the complex (`t_INT`, `t_REAL`, `t_FRAC` or `t_COMPLEX`) x to a real number if its imaginary part is 0. Shallow function.

converts the `t_COMPLEX` `x` to a a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`.

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of precision `prec`; not necessarily of *length* `prec`: a real 0 may be given as `real_0(...)`). If the result is a `t_COMPLEX` extra care is taken so that its modulus really has accuracy `prec`: there is a problem if the real part of the input is an exact 0; indeed, converting it to `real_0(prec)` would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in $1.E-100+0.E-38 = 0.E-38$.

`GEN gtomp(GEN z, long prec)` converts the real number `x` (`t_INT`, `t_REAL`, `t_FRAC`, real `t_QUAD`) to either a `t_INT` or a `t_REAL` of precision `prec`. Not stack-clean if x is a `t_INT`: we return x itself and not a copy.

`GEN gcvtop(GEN x, GEN p, long l)` converts x into a `t_PADIC` of precision l. Works componentwise on recursive objects, e.g. `t_POL` or `t_VEC`. Converting 0 yields $O(p^l)$; converting a nonzero number yield a result well defined modulo $p^{v_p(x)+l}$.

`GEN cvtop(GEN x, GEN p, long l)` as `gcvtop`, assuming that x is a scalar.

`GEN cvtop2(GEN x, GEN y)` y being a p-adic, converts the scalar x to a p-adic of the same accuracy. Shallow function.

`GEN cvstop2(long s, GEN y)` y being a p-adic, converts the scalar s to a p-adic of the same accuracy. Shallow function.

`GEN gprec(GEN x, long l)` returns a copy of x whose precision is changed to l digits. The precision change is done recursively on all components of x. Digits means *decimal*, p-adic and X-adic digits for `t_REAL`, `t_SER`, `t_PADIC` components, respectively.

`GEN gprec_w(GEN x, long prec)` returns a shallow copy of x whose `t_REAL` components have their precision changed to `prec` bits. This is often more useful than `gprec`.

`GEN gprec_wtrunc(GEN x, long prec)` returns a shallow copy of x whose `t_REAL` components have their precision *truncated* to `prec` bits. Contrary to `gprec_w`, this function may never increase the precision of x.

`GEN gprec_wensure(GEN x, long prec)` returns a shallow copy of x whose `t_REAL` components have their precision *increased* to at least *prec* bits. Contrary to `gprec_w`, this function may never decrease the precision of x.

The following functions are obsolete and kept for backward compatibility only:

`GEN precision0(GEN x, long n)`

`GEN bitprecision0(GEN x, long n)`

9.1.2 Modular objects / lifts.

`GEN gmodulo(GEN x, GEN y)` creates the object **Mod**`(x,y)` on the PARI stack, where `x` and `y` are either both `t_INT`s, and the result is a `t_INTMOD`, or `x` is a scalar or a `t_POL` and `y` a `t_POL`, and the result is a `t_POLMOD`.

`GEN gmodulgs(GEN x, long y)` same as **gmodulo** except `y` is a `long`.

`GEN gmodulsg(long x, GEN y)` same as **gmodulo** except `x` is a `long`.

`GEN gmodulss(long x, long y)` same as **gmodulo** except both `x` and `y` are `long`s.

`GEN lift_shallow(GEN x)` shallow version of `lift`

`GEN liftall_shallow(GEN x)` shallow version of `liftall`

`GEN liftint_shallow(GEN x)` shallow version of `liftint`

`GEN liftpol_shallow(GEN x)` shallow version of `liftpol`

`GEN centerlift0(GEN x, long v)` DEPRECATED, kept for backward compatibility only: use either `lift0`(x, v) or `centerlift`(x).

9.1.3 Between polynomials and coefficient arrays.

`GEN gtopoly(GEN x, long v)` converts or truncates the object `x` into a `t_POL` with main variable number `v`. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. `[2,3]` goes to `2*v + 3`

`GEN gtopolyrev(GEN x, long v)` converts or truncates the object `x` into a `t_POL` with main variable number `v`, but vectors are converted in reverse order compared to `gtopoly` (coefficients are given by increasing degree). E.g. `[2,3]` goes to `3*v + 2`. In other words the vector represents a polynomial in the basis $(1, v, v^2, v^3, \ldots)$.

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. For internal use. Normalizing means deleting all leading *exact* zeroes (as per `isexactzero`), except if the polynomial turns out to be 0, in which case we try to find a coefficient c which is a nonrational zero, and return the constant polynomial c. (We do this so that information about the base ring is not lost.)

`GEN normalizepol_lg(GEN x, long l)` applies `normalizepol` to `x`, pretending that `lg(x)` is l, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `normalizepol(x)`.

`GEN normalizepol_approx(GEN x, long lx)` as `normalizepol_lg`, with the difference that we just delete all leading zeroes (as per `gequal0`). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by x.

The following routines are all shallow: they do *not* copy coefficients on the stack and only move pointers around, hence are very fast but not suitable for `gc_GEN_unsafe` or `gc_upto` calls. Recall that an `RgV` (resp. an `RgX`, resp. an `RgM`) is a `t_VEC` or `t_COL` (resp. a `t_POL`, resp. a `t_MAT`) with arbitrary components. Similarly, an `RgXV` is a `t_VEC` or `t_COL` with `RgX` components, etc.

`GEN RgV_to_RgX(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopolyrev`, without copy).

`GEN RgV_to_RgX_reverse(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopoly`, without copy).

GEN RgX_to_RgC(GEN x, long N) converts the t_POL x to a t_COL v with N components. Coefficients of x are listed by increasing degree, so that y[i] is the coefficient of the term of degree $i-1$ in x.

GEN Rg_to_RgC(GEN x, long N) as RgX_to_RgV, except that other types than t_POL are allowed for x, which is then considered as a constant polynomial.

GEN RgM_to_RgXV(GEN x, long v) converts the RgM x to a t_VEC of RgX, by repeated calls to RgV_to_RgX.

GEN RgM_to_RgXV_reverse(GEN x, long v) converts the RgM x to a t_VEC of RgX, by repeated calls to RgV_to_RgX_reverse.

GEN RgV_to_RgM(GEN v, long N) converts the vector v to a t_MAT with N rows, by repeated calls to Rg_to_RgV.

GEN RgXV_to_RgM(GEN v, long N) converts the vector of RgX v to a t_MAT with N rows, by repeated calls to RgX_to_RgV.

GEN RgM_to_RgXX(GEN x, long v, long w) converts the RgM x into a t_POL in variable v, whose coefficients are t_POLs in variable w. This is a shortcut for

```
RgV_to_RgX( RgM_to_RgXV(x, w), v );
```

There are no consistency checks with respect to variable priorities: the above is an invalid object if varncmp(v, w) ≥ 0.

GEN RgXX_to_RgM(GEN x, long N) converts the t_POL x with RgX (or constant) coefficients to a matrix with N rows.

long RgXY_degreex(GEN P) return the degree of P with respect to the secondary variable.

GEN RgXY_derivx(GEN P) return the derivative of P with respect to the secondary variable.

GEN RgXY_swap(GEN P, long n, long w) converts the bivariate polynomial P(u, v) (a t_POL with t_POL or scalar coefficients) to $P($pol_x[w]$, u)$, assuming n is an upper bound for $\deg_v($P$)$.

GEN RgXY_swapspec(GEN C, long n, long w, long lP) as RgXY_swap where the coefficients of P are given by gel(C,0),...,gel(C,lP-1).

GEN RgX_to_ser(GEN x, long l) convert the t_POL x to a *shallow* t_SER of length $l \geq 2$. Unless the polynomial is an exact zero, the coefficient of lowest degree T^d of the result is not an exact zero (as per isexactzero). The remainder is $O(T^{d+l-2})$.

GEN RgX_to_ser_inexact(GEN x, long l) convert the t_POL x to a *shallow* t_SER of length $l \geq 2$. Unless the polynomial is zero, the coefficient of lowest degree T^d of the result is not zero (as per gequal0). The remainder is $O(T^{d+l-2})$.

GEN RgV_to_ser(GEN x, long v, long l) convert the t_VEC x, to a *shallow* t_SER of length $l \geq 2$.

GEN rfrac_to_ser(GEN F, long l) applied to a t_RFRAC F, creates a t_SER of length $l \geq 2$ congruent to F. Not stack-clean but suitable for gc_upto.

GEN rfrac_to_ser_i(GEN F, long l) internal variant of rfrac_to_ser. Shallow function.

GEN rfracrecip_to_ser_absolute(GEN F, long d) applied to a t_RFRAC F, creates the t_SER $F(1/t) + O(t^d)$. Note that we use absolute and not relative precision here.

`GEN gtoser(GEN s, long v, long d)`. This function is deprecated, kept for backward compatibility: it follows the semantic of `Ser(s,v)`, with $d =$ `seriesprecision` implied and is hard to use as a general conversion function. Use `gtoser_prec` instead.

It converts the object s into a `t_SER` with main variable number `v` and $d > 0$ significant terms, but the argument d is sometimes ignored. More precisely

• if s is a scalar (with respect to variable v), we return a constant power series with d significant terms;

• if s is a `t_POL` in variable v, it is truncated to d terms if needed;

• if s is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision d is *ignored*;

• if s is already a power series in v, we return a copy, and the precision d is again *ignored*.

`GEN gtoser_prec(GEN s, long v, long d)` this function is a variant of `gtoser` following the semantic of `Ser(s,v,d)`: the precision d is always taken into account.

`GEN gtocol(GEN x)` converts the object `x` into a `t_COL`

`GEN gtomat(GEN x)` converts the object `x` into a `t_MAT`.

`GEN gtovec(GEN x)` converts the object `x` into a `t_VEC`.

`GEN gtovecsmall(GEN x)` converts the object `x` into a `t_VECSMALL`.

`GEN normalizeser(GEN x)` applied to an unnormalized `t_SER` `x` (i.e. type `t_SER` with all coefficients correctly set except that `x[2]` might be zero), normalizes `x` correctly in place. Returns `x`. For internal use.

`GEN serchop0(GEN s)` given a `t_SER` of the form $x^v s(x)$, with $s(0) \neq 0$, return $x^v(s - s(0))$. Shallow function.

`GEN serchop_i(GEN x, long n)` returns a shallow chopy of `t_SER` x with all terms of degree strictly less than n removed. Shallow version of `serchop`.

9.2 Constructors.

9.2.1 Clean constructors.

`GEN zeropadic(GEN p, long n)` creates a 0 `t_PADIC` equal to $O(\mathtt{p}^{\mathtt{n}})$.

`GEN zeroser(long v, long n)` creates a 0 `t_SER` in variable `v` equal to $O(X^{\mathtt{n}})$.

`GEN scalarser(GEN x, long v, long prec)` creates a constant `t_SER` in variable `v` and precision `prec`, whose constant coefficient is (a copy of) `x`, in other words $\mathtt{x} + O(\mathtt{v}^{\mathtt{prec}})$. Assumes that $\mathtt{prec} \geq 0$.

`GEN pol_0(long v)` Returns the constant polynomial 0 in variable v.

`GEN pol_1(long v)` Returns the constant polynomial 1 in variable v.

`GEN pol_x(long v)` Returns the monomial of degree 1 in variable v.

`GEN pol_xn(long n, long v)` Returns the monomial of degree n in variable v; assume that $n \geq 0$.

`GEN pol_xnall(long n, long v)` Returns the Laurent monomial of degree n in variable v; $n < 0$ is allowed.

`GEN pol_x_powers(long N, long v)` returns the powers of `pol_x(v)`, of degree 0 to $N-1$, in a vector with N components.

`GEN scalarpol(GEN x, long v)` creates a constant `t_POL` in variable `v`, whose constant coefficient is (a copy of) `x`.

`GEN deg1pol(GEN a, GEN b, long v)` creates the degree 1 `t_POL` a`pol_x`$(v) + b$

`GEN zeropol(long v)` is identical `pol_0`.

`GEN zerocol(long n)` creates a `t_COL` with `n` components set to `gen_0`.

`GEN zerovec(long n)` creates a `t_VEC` with `n` components set to `gen_0`.

`GEN zerovec_block(long n)` as `zerovec` but return a clone.

`GEN col_ei(long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

`GEN vec_ei(long n, long i)` creates a `t_VEC` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

`GEN trivial_fact(void)` returns the trivial (empty) factorization `Mat([]~,[]~)`

`GEN prime_fact(GEN x)` returns the factorization `Mat([x]~, [1]~)`

`GEN Rg_col_ei(GEN x, long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `x`.

`GEN vecsmall_ei(long n, long i)` creates a `t_VECSMALL` with `n` components set to `0`, but for the `i`-th one which is set to `1` (`i`-th vector in the canonical basis).

`GEN scalarcol(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a copy of `x`. (The name comes from `RgV_isscalar`.)

`GEN mkintmodu(ulong x, ulong y)` creates the `t_INTMOD Mod(x, y)`. The inputs must satisfy $x < y$.

`GEN zeromat(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use `zeromatcopy`.

`GEN zeromatcopy(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`.

`GEN matid(long n)` identity matrix in dimension `n` (with components `gen_1` and`gen_0`).

`GEN scalarmat(GEN x, long n)` scalar matrix, `x` times the identity.

`GEN scalarmat_s(long x, long n)` scalar matrix, `stoi(x)` times the identity.

`GEN vecrange(GEN a, GEN b)` returns the `t_VEC` $[a..b]$.

`GEN vecrangess(long a, long b)` returns the `t_VEC` $[a..b]$.

See also next section for analogs of the following functions:

`GEN mkfracss(long x, long y)` creates the `t_FRAC` x/y. Assumes that $y > 1$ and $(x, y) = 1$.

`GEN sstoQ(long x, long y)` returns the `t_INT` or `t_FRAC` x/y; no assumptions.

`GEN uutoQ(ulong x, ulong y)` returns the `t_INT` or `t_FRAC` x/y; no assumptions.

`void Qtoss(GEN q, long *n, long *d)` given a `t_INT` or `t_FRAC` q, set n and d such that $q = n/d$ with $d \geq 1$ and $(n, d) = 1$. Overflow error if numerator or denominator do not fit into a long integer.

`GEN mkfraccopy(GEN x, GEN y)` creates the `t_FRAC` x/y. Assumes that $y > 1$ and $(x, y) = 1$.

`GEN mkrfraccopy(GEN x, GEN y)` creates the `t_RFRAC` x/y. Assumes that y is a `t_POL`, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

`GEN mkcolcopy(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

`GEN mkmatcopy(GEN x)` creates a 1-by-1 `t_MAT` wrapping the `t_COL x`.

`GEN mkveccopy(GEN x)` creates a 1-dimensional `t_VEC` containing `x`.

`GEN mkvec2copy(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to `[x,y]`.

`GEN mkcols(long x)` creates a 1-dimensional `t_COL` containing `stoi(x)`.

`GEN mkcol2s(long x, long y)` creates a 2-dimensional `t_COL` containing `[stoi(x), stoi(y)]` .

`GEN mkcol3s(long x, long y, long z)` creates a 3-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z)]` .

`GEN mkcol4s(long x, long y, long z, long t)` creates a 4-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]` .

`GEN mkvecs(long x)` creates a 1-dimensional `t_VEC` containing `stoi(x)`.

`GEN mkvec2s(long x, long y)` creates a 2-dimensional `t_VEC` containing `[stoi(x), stoi(y)]`.

`GEN mkmat22s(long a, long b, long c, long d)` creates the 2 by 2 `t_MAT` with successive rows `[stoi(a), stoi(b)]` and `[stoi(c), stoi(d)]`.

`GEN mkvec3s(long x, long y, long z)` creates a 3-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z)]`.

`GEN mkvec4s(long x, long y, long z, long t)` creates a 4-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]`.

`GEN mkvec5s(long x, long y, long z, long t, long u)` creates a 5-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t), stoi(u)]`.

`GEN mkvec6s(long x, long y, long z, long t, long u, long v)` creates a 6-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t), stoi(u), stoi(v)]`.

`GEN mkvecsmall(long x)` creates a 1-dimensional `t_VECSMALL` containing `x`.

`GEN mkvecsmall2(long x, long y)` creates a 2-dimensional `t_VECSMALL` containing `[x, y]`.

`GEN mkvecsmall3(long x, long y, long z)` creates a 3-dimensional `t_VECSMALL` containing `[x, y, z]`.

`GEN mkvecsmall4(long x, long y, long z, long t)` creates a 4-dimensional `t_VECSMALL` containing `[x, y, z, t]`.

`GEN mkvecsmall5(long x, long y, long z, long t, long u)` creates a 5-dimensional `t_VECSMALL` containing `[x, y, z, t, u]`.

`GEN mkvecsmalln(long n, ...)` returns the `t_VECSMALL` whose n coefficients (`long`) follow.

Warning.. Since this is a variadic function, the compiler does not perform C type promotion on the arguments. You have to ensure that all the arguments following n are of type `long`, in particular integer constants need to be written with the `L` suffix: `mkvecsmalln(2, 1L, 2L)` is correct, but `mkvecsmalln(2, 1, 2)` is not.

9.2.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for `gc_upto`. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g `gen_0`), then the above warning does not apply.

`GEN mkcomplex(GEN x, GEN y)` creates the `t_COMPLEX` $x + iy$.

`GEN mulcxI(GEN x)` creates the `t_COMPLEX` ix. The result in general contains data pointing back to the original x. Use `gcopy` if this is a problem. But in most cases, the result is to be used immediately, before x is subject to garbage collection.

`GEN mulcxmI(GEN x)`, as `mulcxI`, but returns $-ix$.

`GEN mulcxpowIs(GEN x, long k)`, as `mulcxI`, but returns $x \cdot i^k$.

`GEN mkquad(GEN n, GEN x, GEN y)` creates the `t_QUAD` $x + yw$, where w is a root of n, which is of the form `quadpoly(D)`.

`GEN quadpoly_i(GEN D)` creates the canonical quadratic polynomial of discriminant D. Assume that the `t_INT` D is congruent to $0, 1 \bmod 4$ and not a square.

`GEN mkfrac(GEN x, GEN y)` creates the `t_FRAC` x/y. Assumes that $y > 1$ and $(x, y) = 1$.

`GEN mkrfrac(GEN x, GEN y)` creates the `t_RFRAC` x/y. Assumes that y is a `t_POL`, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

`GEN mkcol(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

`GEN mkcol2(GEN x, GEN y)` creates a 2-dimensional `t_COL` equal to `[x,y]`.

`GEN mkcol3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_COL` equal to `[x,y,z]`.

`GEN mkcol4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_COL` equal to `[x,y,z,t]`.

`GEN mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_COL` equal to $[a_1, a_2, a_3, a_4, a_5]$.

`GEN mkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` creates the 6-dimensional column vector `[x,y,z,t,u,v]` .

`GEN mkintmod(GEN x, GEN y)` creates the `t_INTMOD Mod(x, y)`. The inputs must be `t_INT`s satisfying $0 \leq x < y$.

`GEN mkpolmod(GEN x, GEN y)` creates the `t_POLMOD Mod(x, y)`. The input must satisfy $\deg x < \deg y$ with respect to the main variable of the `t_POL` y. x may be a scalar.

`GEN mkpadic(GEN u, GEN p, GEN pd, long v, long d)` creates the `t_PADIC` $p^v(u + O(p^d))$ of valuation v and precision d. The arguments $u, p,$ `pd` must be `t_INT` with $0 \leq u < p^d$, p prime and `pd` $= p^d$. The value $u = 0$ is allowed if and only if $d = 0$, otherwise u must be a p-unit.

`GEN mkmat(GEN x)` creates a 1-column `t_MAT` with column x (a `t_COL`).

`GEN mkmat2(GEN x, GEN y)` creates a 2-column `t_MAT` with columns x, y (`t_COL`s of the same length).

`GEN mkmat22(GEN a, GEN b, GEN c, GEN d)` creates the 2 by 2 `t_MAT` with successive rows $[a, b]$ and $[c, d]$.

`GEN mkmat3(GEN x, GEN y, GEN z)` creates a 3-column `t_MAT` with columns x, y, z (`t_COL`s of the same length).

`GEN mkmat4(GEN x, GEN y, GEN z, GEN t)` creates a 4-column `t_MAT` with columns x, y, z, t (`t_COL`s of the same length).

`GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` creates a 5-column `t_MAT` with columns x, y, z, t, u (`t_COL`s of the same length).

`GEN mkvec(GEN x)` creates a 1-dimensional `t_VEC` containing `x`.

`GEN mkvec2(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to `[x,y]`.

`GEN mkvec3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_VEC` equal to `[x,y,z]`.

`GEN mkvec4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_VEC` equal to `[x,y,z,t]`.

`GEN mkvec5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_VEC` equal to $[a_1, a_2, a_3, a_4, a_5]$.

`GEN mkvec6(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5, GEN a6)` creates the 6-dimensional `t_VEC` equal to $[a_1, a_2, a_3, a_4, a_5, a_6]$.

`GEN mkqfb(GEN a, GEN b, GEN c, GEN D)` creates `t_QFB` equal to `Qfb(a,b,c)`, assuming that $D = b^2 - 4ac$.

`GEN mkerr(long n)` returns a `t_ERROR` with error code n (`enum err_list`).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for `gc_upto`. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

`GEN retmkvec(GEN x)` returns a vector containing the single entry x, where the vector root is created just before the function argument x is evaluated. Expands to

```
{
  GEN res = cgetg(2, t_VEC);
  gel(res, 1) = x; /*  or rather, the expansion of x */
  return res;
}
```

For instance, the `retmkvec(gcopy(x))` returns a clean object, just like `return mkveccopy(x)` would.

`GEN retmkvec2(GEN x, GEN y)` returns the 2-dimensional `t_VEC [x,y]`.

`GEN retmkvec3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_VEC [x,y,z]`.

`GEN retmkvec4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_VEC [x,y,z,t]`.

`GEN retmkvec5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional row vector `[x,y,z,t,u]`.

`GEN retmkvec6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` returns the 5-dimensional row vector `[x,y,z,t,u,v]`.

`GEN retconst_vec(long n, GEN x)` returns the n-dimensional `t_VEC` whose entries are constant and all equal to x.

`GEN retmkcol(GEN x)` returns the 1-dimensional `t_COL [x]` .

`GEN retmkcol2(GEN x, GEN y)` returns the 2-dimensional `t_COL [x,y]` .

`GEN retmkcol3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_COL [x,y,z]` .

`GEN retmkcol4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_COL [x,y,z,t]` .

`GEN retmkcol5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional column vector `[x,y,z,t,u]` .

`GEN retmkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` returns the 6-dimensional column vector `[x,y,z,t,u,v]` .

`GEN retconst_col(long n, GEN x)` returns the n-dimensional `t_COL` whose entries are constant and all equal to x.

`GEN retmkmat(GEN x)` returns the 1-column `t_MAT` with colum `x`.

`GEN retmkmat2(GEN x, GEN y)` returns the 2-column `t_MAT` with columns `x`, `y`.

`GEN retmkmat3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_MAT` with columns `x`, `y`, `z`.

`GEN retmkmat4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`.

`GEN retmkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`, `u`.

`GEN retmkmat22(GEN a, GEN b, GEN c, GEN d)` return the 2 by 2 `t_MAT` with successive rows $[a, b]$ and $[c, d]$.

`GEN retmkcomplex(GEN x, GEN y)` returns the `t_COMPLEX` `x + I*y`.

`GEN retmkfrac(GEN x, GEN y)` returns the `t_FRAC` `x / y`. Assume x and y are coprime and $y > 1$.

`GEN retmkrfrac(GEN x, GEN y)` returns the `t_RFRAC` `x / y`. Assume x and y are coprime and more generally that the rational function cannot be simplified.

`GEN retmkintmod(GEN x, GEN y)` returns the `t_INTMOD` `Mod(x, y)`.

`GEN retmkpadic(GEN u, GEN p, GEN pd, long v, long d)` returns the `t_PADIC` $p^v(u + O(p^d))$ of valuation v and precision d. See `mkpadic` for constraints on arguments.

`GEN retmkpadic_i(GEN u, GEN p, GEN pd, long v, long d)` this is a technical internal variant of `retmkpadic`, where the `pd` argument is an expression computing `_pd` $:= p^d$ and the u argument is an expression allowed to use that private `_pd` variable. Both expressions must be stack-clean. This is used as in

```
retmkpadic_i(modii(x, _pd), p, powiu(p,d), 0, d);
```

where the `_pd` trick allows to defer evaluation until the macro has created the `t_PADIC` root of the object. The result is then stack-clean.

`GEN retmkquad(GEN n, GEN a, GEN b)`.

`GEN retmkpolmod(GEN x, GEN y)` returns the `t_POLMOD` `Mod(x, y)`.

`GEN retmkqfb(GEN a, GEN b, GEN c, GEN D)` returns the `t_QFB` equal to `Qfb(a,b,c)`, assuming that $D = b^2 - 4ac$.

`GEN mkintn(long n, ...)` returns the nonnegative `t_INT` whose expansion in base 2^{32} is given by the following n digits. Digits are `unsigned int` less than 2^{32}.

```
mkintn(3, 0x1U, 0x2U, 0x3U);
```

returns $2^{64} + 2 \cdot 2^{32} + 3$.

Warning.. Since this is a variadic function, the compiler does not perform C type promotion on the arguments. You have to ensure that all the arguments following n are of type `unsigned int`, in particular integer constants need to be written with the `U` suffix as above: `mkvecsmalln(3, 1, 2, 3)` would not be correct.

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

`GEN scalarcol_shallow(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a shallow copy of `x`. (The name comes from `RgV_isscalar`.)

`GEN scalarmat_shallow(GEN x, long n)` creates an $n \times n$ scalar matrix whose diagonal is set to shallow copies of the scalar `x`.

`GEN RgX_sylvestermatrix(GEN f, GEN g)` return the Sylvester matrix attached to the two `t_POL` in the same variable f and g.

`GEN diagonal_shallow(GEN x)` returns a diagonal matrix whose diagonal is given by the vector x. Shallow function.

`GEN scalarpol_shallow(GEN a, long v)` returns the degree 0 `t_POL` a`pol_x`$(v)^0$.

`GEN deg1pol_shallow(GEN a, GEN b, long v)` returns the degree 1 `t_POL` a`pol_x`$(v) + b$

`GEN deg2pol_shallow(GEN a, GEN b, GEN c, long v)` returns the degree 2 `t_POL` $ax^2 + bx + c$ where $x =$ `pol_x`(v).

`GEN zeropadic_shallow(GEN p, long n)` returns a (shallow) 0 `t_PADIC` equal to $O($`p`$^{\text{n}})$.

9.2.3 From roots to polynomials.

`GEN deg1_from_roots(GEN L, long v)` given a vector L of scalars, returns the vector of monic linear polynomials in variable v whose roots are the $L[i]$, i.e. the $x - L[i]$.

`GEN roots_from_deg1(GEN L)` given a vector L of monic linear polynomials, return their roots, i.e. the $-L[i](0)$.

`GEN roots_to_pol(GEN L, long v)` given a vector of scalars L, returns the monic polynomial in variable v whose roots are the $L[i]$. Leaves some garbage on stack, but suitable for `gc_upto`.

`GEN roots_to_pol_r1(GEN L, long v, long r1)` as `roots_to_pol` assuming the first r_1 roots are "real", and the following ones are representatives of conjugate pairs of "complex" roots. So if L has $r_1 + r_2$ elements, we obtain a polynomial of degree $r_1 + 2r_2$. In most applications, the roots are indeed real and complex, but the implementation assumes only that each "complex" root z introduces a quadratic factor $X^2 - \texttt{trace}(z)X + \texttt{norm}(z)$. Leaves some garbage on stack, but suitable for `gc_upto`.

9.3 Integer parts.

`GEN gfloor(GEN x)` creates the floor of `x`, i.e. the (true) integral part.

`GEN gfrac(GEN x)` creates the fractional part of `x`, i.e. `x` minus the floor of `x`.

`GEN gceil(GEN x)` creates the ceiling of `x`.

`GEN ground(GEN x)` rounds towards $+\infty$ the components of `x` to the nearest integers.

`GEN grndtoi(GEN x, long *e)` same as `ground`, but in addition sets `*e` to the binary exponent of $x - \texttt{ground}(x)$. If this is positive, then significant bits are lost in the rounded result. This kind of situation raises an error message in **ground** but not in **grndtoi**. The parameter e can be set to `NULL` if an error estimate is not needed, for a minor speed up.

`GEN gtrunc(GEN x)` truncates `x`. This is the false integer part if `x` is a real number (i.e. the unique integer closest to `x` among those between 0 and `x`). If `x` is a `t_SER`, it is truncated to a `t_POL`; if `x` is a `t_RFRAC`, this takes the polynomial part.

`GEN gtrunc2n(GEN x, long n)` creates the floor of 2^n`x`, this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

`GEN gcvtoi(GEN x, long *e)` analogous to **grndtoi** for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

9.4 Valuation and shift.

`GEN gshift(GEN x, long n)` yields the result of shifting (the components of) `x` left by `n` (if `n` is nonnegative) or right by $-$`n` (if `n` is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by $2^{\texttt{n}}$.

`GEN gmul2n(GEN x, long n)` yields the product of `x` and $2^{\texttt{n}}$. This is different from `gshift` when `n` is negative and `x` is a `t_INT`: **gshift** truncates, while **gmul2n** creates a fraction if necessary.

`long gvaluation(GEN x, GEN p)` returns the greatest exponent e such that `p`^{e} divides `x`, when this makes sense.

`long gval(GEN x, long v)` returns the highest power of the variable number `v` dividing the `t_POL` `x`.

9.5 Comparison operators.

9.5.1 Generic.

`long gcmp(GEN x, GEN y)` comparison of `x` with `y`: returns 1 ($x > y$), 0 ($x = y$) or -1 ($x < y$). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` cannot be compared to any non-string type. If neither x nor y is a `t_STR`, their allowed types are `t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` with positive discriminant (use the canonical embedding $w \to \sqrt{D}/2$ or $w \to (1 + \sqrt{D})/2$) or `t_INFINITY`. Use `cmp_universal` to compare arbitrary `GEN`s.

`long lexcmp(GEN x, GEN y)` comparison of `x` with `y` for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp`(x, y) if both arguments are scalars.

`int gequalX(GEN x)` return 1 (true) if `x` is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

`long gequal(GEN x, GEN y)` returns 1 (true) if `x` is equal to `y`, 0 otherwise. A priori, this makes sense only if `x` and `y` have the same type, in which case they are recursively compared componentwise. When the types are different, a `true` result means that `x - y` was successfully computed and that `gequal0` found it equal to 0. In particular

```
gequal(cgetg(1, t_VEC), gen_0)
```

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

`long gidentical(GEN x, GEN y)` returns 1 (true) if `x` is identical to `y`, 0 otherwise. In particular, the types and length of `x` and `y` must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

`GEN gmax(GEN x, GEN y)` returns a copy of the maximum of x and y, compared using `gcmp`.

`GEN gmin(GEN x, GEN y)` returns a copy of the minimum of x and y, compared using `gcmp`.

`GEN gmax_shallow(GEN x, GEN y)` shallow version of `gmax`.

`GEN gmin_shallow(GEN x, GEN y)` shallow version of `gmin`.

9.5.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if `x` is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`GEN gisexactzero(GEN x)` returns `NULL` unless `x` is exactly equal to 0 (as per `isexactzero`). When `x` is an exact zero return the attached scalar zero as a `t_INT` (`gen_0`), a `t_INTMOD` (`Mod(0,`N`)` for the largest possible N) or a `t_FFELT`.

`int isrationalzero(GEN x)` returns 1 (true) if `x` is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzero(GEN x)` returns 1 (true) if `x` is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to -1.

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to -1, and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *nonzero* `t_INT`, return 1 (true) if `x` is equal to -1 or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to -1, 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gmings(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

9.6 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

```
is_scalar_t(typ(x)) && isrationalzero(x)
```

`int isinexact(GEN x)` returns 1 (true) if x has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if x has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if x is approximately real to the bit accuracy e, and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part t satisfying `gexpo`$(t) < e$.

`int isint(GEN x, GEN *n)` returns 0 (false) if `x` does not round to an integer. Otherwise, returns 1 (true) and set `n` to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if `x` does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set `n` to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if `x` is a complex number (of component types embeddable into the reals) but is not itself real, 0 if `x` is a real (not necessarily of type `t_REAL`), or raises an error if `x` is not embeddable into the complex numbers.

9.6.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

`GEN gle(GEN x, GEN y)`

`GEN glt(GEN x, GEN y)`

`GEN gge(GEN x, GEN y)`

`GEN ggt(GEN x, GEN y)`

`GEN geq(GEN x, GEN y)`

`GEN gne(GEN x, GEN y)`

`GEN gor(GEN x, GEN y)`

`GEN gand(GEN x, GEN y)`

`GEN gnot(GEN x, GEN y)`

9.7 Sorting.

9.7.1 Basic sort.

`GEN sort(GEN x)` sorts the vector `x` in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of x, use `gen_sort_inplace` for a more memory-efficient function.

`GEN lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN vecsort(GEN x, GEN k)`, as `sort`, but sorts the vector `x` in ascending *lexicographic* order, according to the entries of the `t_VECSMALL k`. For example, if $\texttt{k} = [2, 1, 3]$, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

9.7.2 Indirect sorting.

`GEN indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to `x`, would sort the vector. The result is a `t_VECSMALL`.

`GEN indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN indexvecsort(GEN x, GEN k)`, as `vecsort`, but only returns the permutation that would sort the vector `x`.

`long vecindexmin(GEN x)` returns the index for a minimal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

9.7.3 Generic sort and search. The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if $x < y$, a positive one if $x > y$ and 0 if $x = y$. The `data` argument is there in case your `cmp` requires additional context.

`GEN gen_sort(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `sort`, with an explicit comparison routine.

`GEN gen_sort_shallow(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, shallow variant of `gen_sort`.

`GEN gen_sort_uniq(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `gen_sort`, removing duplicate entries.

`GEN gen_indexsort(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`.

`GEN gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`, removing duplicate entries.

`void gen_sort_inplace(GEN x, void *data, int (*cmp)(void*, GEN, GEN), GEN *perm)` sort `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN gen_setminus(GEN A, GEN B, int (*cmp)(GEN, GEN))` given two sorted vectors A and B, returns the vector of elements of A not belonging to B.

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *, GEN, GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x, GEN y, void *data, int (*cmp)(void *, GEN, GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed. Shallow function.

`GEN setunion_i(GEN x, GEN y)` shallow version of `setunion`, a simple alias for

```
merge_sort_uniq(x,y, (void*)cmp_universal, cmp_nodata)
```

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *, GEN, GEN))` let `fx` and `fy` be factorization matrices for X and Y sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of $X * Y$.

`GEN ZM_merge_factor(GEN fx, GEN fy)` as `merge_factor`, where `fx` and `fy` are factorization matrices of integers.

`long gen_search(GEN v, GEN y, void *data, int (*cmp)(void*, GEN, GEN))`.
Let v be a vector sorted according to `cmp(data,a,b)`; look for an index i such that $v[i]$ is equal to y. If y is found, return i (not necessarily the first occurence in case of multisets), else return $-i$ where i is the index where y should be inserted.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN, GEN))` is a faster implementation for the common case `gen_search(T,x,cmp,cmp_nodata)` when we have no need to insert missing elements; return 0 in case x is not found.

9.7.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int cmp_Flx(GEN x, GEN y)` compare two `Flx`, which we assume have the same main variable (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same nf. Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

`int cmp_padic(GEN x, GEN y)` compare two `t_PADIC` (for the same prime p).

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int(*)(GEN,GEN)) data;
```

9.8 Division.

`GEN gdivgu(GEN x, ulong u)` return x/u).

`GEN gdivgunextu(GEN x, ulong u)` return $x/(u(u+1))$. If $u(u+1)$ does not fit into an `ulong`, it is created and left on the stack for efficiency.

`GEN divrunextu(GEN x, ulong i)` as `gdivgunextu` for a `t_REAL` x.

9.9 Divisibility, Euclidean division.

`GEN gdivexact(GEN x, GEN y)` returns the quotient `x/y`, assuming `y` divides `x`. Not stack clean if $y = 1$ (we return x, not a copy).

`int gdvd(GEN x, GEN y)` returns 1 (true) if `y` divides `x`, 0 otherwise.

`GEN gdiventres(GEN x, GEN y)` creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of `x` and `y`.

`GEN gdivent(GEN x, GEN y)` yields the true Euclidean quotient of `x` and the `t_INT` or `t_POL` `y`, as per the `\` GP operator.

`GEN gdiventsg(long s, GEN y)`, as `gdivent` except that `x` is a `long`.

`GEN gdiventgs(GEN x, long s)`, as `gdivent` except that `y` is a `long`.

`GEN gmod(GEN x, GEN y)` yields the remainder of `x` modulo the `t_INT` or `t_POL` `y`, as per the `%` GP operator. A `t_REAL` or `t_FRAC` `y` is also allowed, in which case the remainder is the unique real r such that $0 \leq r < |\mathtt{y}|$ and $\mathtt{y} = q\mathtt{x} + r$ for some (in fact unique) integer q.

`GEN gmodsg(long s, GEN y)` as `gmod`, except `x` is a `long`.

`GEN gmodgs(GEN x, long s)` as `gmod`, except `y` is a `long`.

`GEN gdivmod(GEN x, GEN y, GEN *r)` If `r` is not equal to `NULL` or `ONLY_REM`, creates the (false) Euclidean quotient of `x` and `y`, and puts (the address of) the remainder into `*r`. If `r` is equal to `NULL`, do not create the remainder, and if `r` is equal to `ONLY_REM`, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`.

`GEN poldivrem(GEN x, GEN y, GEN *r)` same as **gdivmod** but specifically for `t_POL`s `x` and `y`, not necessarily in the same variable. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN gdeuc(GEN x, GEN y)` creates the Euclidean quotient of the `t_POL`s `x` and `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN grem(GEN x, GEN y)` creates the Euclidean remainder of the `t_POL` `x` divided by the `t_POL` `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

`GEN gdivround(GEN x, GEN y)` if `x` and `y` are real (`t_INT`, `t_REAL`, `t_FRAC`), return the rounded Euclidean quotient of x and y as per the `\/` GP operator. Operate componentwise if `x` is a `t_COL`, `t_VEC` or `t_MAT`. Otherwise as **gdivent**.

`GEN centermod_i(GEN x, GEN y, GEN y2)`, as `centermodii`, componentwise.

`GEN centermod(GEN x, GEN y)`, as `centermod_i`, except that `y2` is computed (and left on the stack for efficiency).

`GEN ginvmod(GEN x, GEN y)` creates the inverse of `x` modulo `y` when it exists. `y` must be of type `t_INT` (in which case `x` is of type `t_INT`) or `t_POL` (in which case `x` is either a scalar type or a `t_POL`).

9.10 GCD, content and primitive part.

9.10.1 Generic.

`GEN resultant(GEN x, GEN y)` creates the resultant of the `t_POL`s `x` and `y` computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in $\mathbf{Q}[X]$) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of `x` and `y` may also be scalars (treated as polynomials of degree 0)

`GEN ggcd(GEN x, GEN y)` creates the GCD of `x` and `y`.

`GEN glcm(GEN x, GEN y)` creates the LCM of `x` and `y`.

`GEN gbezout(GEN x, GEN y, GEN *u, GEN *v)` returns the GCD of `x` and `y`, and puts (the addresses of) objects u and v such that $u\mathtt{x} + v\mathtt{y} = \gcd(\mathtt{x}, \mathtt{y})$ into `*u` and `*v`.

`GEN subresext(GEN x, GEN y, GEN *U, GEN *V)` returns the resultant of `x` and `y`, and puts (the addresses of) polynomials u and v such that $u\mathtt{x} + v\mathtt{y} = \mathrm{Res}(\mathtt{x}, \mathtt{y})$ into `*U` and `*V`.

`GEN content(GEN x)` returns the GCD of all the components of `x`.

`GEN primitive_part(GEN x, GEN *c)` sets `c` to `content(x)` and returns the primitive part `x / c`. A trivial content is set to `NULL`.

`GEN primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN denom_i(GEN x)` shallow version of `denom`.

`GEN numer_i(GEN x)` shallow version of `numer`.

9.10.2 Over the rationals.

`long Q_pval(GEN x, GEN p)` valuation at the `t_INT p` of the `t_INT` or `t_FRAC x`.

`long Q_lval(GEN x, ulong p)` same for `ulong` p.

`long Q_pvalrem(GEN x, GEN p, GEN *r)` returns the valuation e at the `t_INT p` of the `t_INT` or `t_FRAC x`. The quotient $\mathtt{x}/\mathtt{p}^e$ is returned in `*r`.

`long Q_lvalrem(GEN x, ulong p, GEN *r)` same for `ulong` p.

`GEN Q_abs(GEN x)` absolute value of the `t_INT` or `t_FRAC x`.

`GEN Qdivii(GEN x, GEN y)`, assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Qdivis(GEN x, long y)`, assuming x is an `t_INT`, return the quotient x/y as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Qdiviu(GEN x, ulong y)`, assuming x is an `t_INT`, return the quotient x/y as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Q_abs_shallow(GEN x)` x being a `t_INT` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(`x`)` otherwise.

`GEN Q_gcd(GEN x, GEN y)` gcd of the `t_INT` or `t_FRAC x` and `y`.

In the following functions, arguments belong to a $M \otimes_{\mathbf{Z}} \mathbf{Q}$ for some natural $\mathbf{Z}$-module M, e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such

objects), and an element of M is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to **Q**, denominators to **Z**. For instance the **Q**-content of $(1/2)xy$ is $(1/2)$, and its **Q**-denominator is 2, whereas `content` would return $y/2$ and `denom` 1.

`GEN Q_content(GEN x)` the **Q**-content of x.

`GEN Z_content(GEN x)` as `Q_content` but assume that all rationals are in fact `t_INT`s and return `NULL` when the content is 1. This function returns as soon as the content is found to equal 1.

`GEN Q_content_safe(GEN x)` as `Q_content`, returning `NULL` when the **Q**-content is not defined (e.g. for a `t_REAL` or `t_INTMOD` component).

`GEN Q_denom(GEN x)` the **Q**-denominator of x. Shallow function. Raises en `e_TYPE` error out when the notion is meaningless, e.g. for a `t_REAL` or `t_INTMOD` component.

`GEN Q_denom_safe(GEN x)` the **Q**-denominator of x. Shallow function. Return `NULL` when the notion is meaningless.

`GEN Q_primitive_part(GEN x, GEN *c)` sets `c` to the **Q**-content of `x` and returns `x / c`, which is integral.

`GEN Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN vec_Q_primpart(GEN x)` as above component-wise. Applied to a `t_MAT`, the result has primitive columns.

`GEN row_Q_primpart(GEN x)` as above, applied to the rows of a `t_MAT`, so that the result has primitive rows. Shallow function.

`GEN Q_remove_denom(GEN x, GEN *ptd)` sets `d` to the **Q**-denominator of `x` and returns `x * d`, which is integral. Shallow function.

`GEN Q_div_to_int(GEN x, GEN c)` returns `x / c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_mul_to_int(GEN x, GEN c)` returns `x * c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_muli_to_int(GEN x, GEN d)` returns `x * c`, assuming c is a `t_INT` and the result is integral.

`GEN mul_content(GEN cx, GEN cy)` `cx` and `cy` are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their product (either a `GEN` or `NULL`).

`GEN div_content(GEN cx, GEN cy)` `cx` and `cy` are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their quotient (either a `GEN` or `NULL`).

`GEN inv_content(GEN c)` c is as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns its inverse (either a `GEN` or `NULL`).

`GEN mul_denom(GEN dx, GEN dy)` `dx` and `dy` are as set by `Q_remove_denom`: either a `t_INT` or `NULL` representing the trivial denominator 1. Returns their product (either a `t_INT` or `NULL`).

9.11 Generic arithmetic operators.

9.11.1 Unary operators.

`GEN gneg(GEN x)` yields $-$`x`.

`GEN gneg_i(GEN x)` shallow function yielding $-$`x`.

`GEN gabs(GEN x)` yields |`x`|.

`GEN gsqr(GEN x)` creates the square of `x`.

`GEN ginv(GEN x)` creates the inverse of `x`.

9.11.2 Binary operators.

Let "*op*" be a binary operation among

op=**add**: addition (`x + y`).

op=**sub**: subtraction (`x - y`).

op=**mul**: multiplication (`x * y`).

op=**div**: division (`x / y`).

The names and prototypes of the functions corresponding to *op* are as follows:

`GEN g`*op*`(GEN x, GEN y)`

`GEN g`*op*`gs(GEN x, long s)`

`GEN g`*op*`gu(GEN x, ulong u)`

`GEN g`*op*`sg(long s, GEN y)`

`GEN g`*op*`ug(ulong u, GEN y)`

Explicitly

`GEN gadd(GEN x, GEN y)`, `GEN gaddgs(GEN x, long s)`, `GEN gaddsg(long s, GEN x)`

`GEN gmul(GEN x, GEN y)`, `GEN gmulgs(GEN x, long s)`, `GEN gmulsg(long s, GEN x)`, `GEN gmulgu(GEN x, ulong u)`, `GEN gmulug(GEN x, ulong u)`,

`GEN gsub(GEN x, GEN y)`, `GEN gsubgs(GEN x, long s)`, `GEN gsubsg(long s, GEN x)`

`GEN gdiv(GEN x, GEN y)`, `GEN gdivgs(GEN x, long s)`, `GEN gdivsg(long s, GEN x)`, `GEN gdivgu(GEN x, ulong u)`,

`GEN gpow(GEN x, GEN y, long l)` creates $\texttt{x}^{\texttt{y}}$. If `y` is a `t_INT`, return `powgi(x,y)` (the precision `l` is not taken into account). Otherwise, the result is $\exp(\texttt{y} * \log(\texttt{x}))$ where exact arguments are converted to floats of precision `l` in case of need; if there is no need, for instance if x is a `t_REAL`, l is ignored. Indeed, if x is a `t_REAL`, the accuracy of $\log x$ is determined from the accuracy of x, it is no problem to multiply by y, even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial $\log x$.

`GEN gpowgs(GEN x, long n)` creates $\texttt{x}^{\texttt{n}}$ using binary powering. To treat the special case $n = 0$, we consider `gpowgs` as a series of `gmul`, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

• gen_1 if x has type t_INT, t_REAL, t_FRAC, or t_PADIC

• Mod(1,N) if x is a t_INTMOD modulo N.

• gen_1 for t_COMPLEX, t_QUAD unless one component is a t_INTMOD, in which case we return Mod(1, N) for a suitable N (the gcd of the moduli that appear).

• FF_1(x) for a t_FFELT.

• qfb_1(x) for t_QFB.

• the identity permutation for t_VECSMALL.

• Rg_get_1(x) otherwise

Of course, the only practical use of this routine for $n = 0$ is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

GEN powgi(GEN x, GEN y) creates x^y, where y is a t_INT, using left-shift binary powering. The case where $y = 0$ (as all cases where y is small) is handled by gpowgs(x, 0).

GEN gpowers(GEN x, long n) returns the vector $[1, x, \ldots, x^n]$.

GEN grootsof1(long n, long prec) returns the vector $[1, x, \ldots, x^{n-1}]$, where x is the n-th root of unity $\exp(2i\pi/n)$.

GEN gsqrpowers(GEN x, long n) returns the vector $[x, x^4, \ldots, x^{n^2}]$.

9.12 Generic operators: product, powering, factorback.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the void* argument provides some arbitrary evaluation context).

GEN gen_product(GEN v, void *D, F2 op) Given two objects x, y, assume that op(D, x, y) implements an associative binary operator. If v has k entries, return

$$v[1]\ op\ v[2]\ op\ \ldots\ op\ v[k];$$

returns gen_1 if $k = 0$ and a copy of $v[1]$ if $k = 1$. Use divide and conquer strategy. Leave some garbage on stack, but suitable for gc_upto if mul is.

GEN gen_pow(GEN x, GEN n, void *D, F1 sqr, F2 mul) $n > 0$ a t_INT, returns x^n; mul(D, x, y) implements the multiplication in the underlying monoid; sqr is a (presumably optimized) shortcut for mul(D, x, x).

GEN gen_powu(GEN x, ulong n, void *D, F1 sqr, F2 mul) $n > 0$, returns x^n. See gen_pow.

GEN gen_pow_i(GEN x, GEN n, void *E, F1 sqr, F2 mul) internal variant of gen_pow, not stack-clean.

`GEN gen_powu_i(GEN x, ulong n, void *E, F1 sqr, F2 mul)` internal variant of `gen_powu`, not stack-clean.

`GEN gen_pow_fold(GEN x, GEN n, void *D, F1 sqr, F1 msqr)` variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr(D, y)` returning xy^2. In particular `D` must implicitly contain x.

`GEN gen_pow_fold_i(GEN x, GEN n, void *E, F1 sqr, F1 msqr)` internal variant of the function `gen_pow_fold`, not stack-clean.

`GEN gen_powu_fold(GEN x, ulong n, void *D, F1 sqr, F1 msqr)`, see `gen_pow_fold`.

`GEN gen_powu_fold_i(GEN x, ulong n, void *E, F1 sqr, F1 msqr)` see `gen_pow_fold_i`.

`GEN gen_pow_init(GEN x, GEN n, long k, void *E, GEN (*sqr)(void*, GEN), GEN (*mul)(void*, GEN, GEN))` Return a table `R` that can be used with `gen_pow_table` to compute the powers of x up to n. The table is of size $2^k \log_2(n)$.

`GEN gen_pow_table(GEN R, GEN n, void *E, GEN (*one)(void*), GEN (*mul)(void*, GEN, GEN))`

Return x^n, where R is as given by `gen_pow_init(x,m,k,E,sqr,mul)` for some integer $m \geq n$.

`GEN gen_powers(GEN x, long n, long usesqr, void *D, F1 sqr, F2 mul, F0 one)` returns $[x^0, \ldots, x^n]$ as a `t_VEC`; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`; `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by x.

`GEN gen_factorback(GEN L, GEN e, void *D, F2 mul, F2 pow, GEN (*one)(void *)D)` generic form of `factorback`. The pair $[L, e]$ is of the form

- `[fa, NULL]`, `fa` a two-column factorization matrix: expand it.
- `[v, NULL]`, v a vector of objects: return their product.
- or `[v, e]`, v a vector of objects, e a vector of integral exponents (a `ZV` or `zv`): return the product of the $v[i]^{e[i]}$.

`mul(D, x, y)` and `pow(D, x, n)` return xy and x^n respectively.

`one(D)` returns the neutral element. If `one` is `NULL`, `gen_1` is used instead.

9.13 Matrix and polynomial norms.

This section concerns only standard norms of $\mathbf{R}$ and $\mathbf{C}$ vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following "container" types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

`GEN gnorml2(GEN x)` The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual L^2 norm. In most applications, the missing square root computation can be skipped.

`GEN gnorml1(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the usual L^1 norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

`GEN gnorml1_fake(GEN x)` as `gnorml1`, except that the norm of a `t_QUAD` $x + wy$ or `t_COMPLEX` $x + Iy$ is defined as $|x| + |y|$, where we use the ordinary real absolute value. This is still a norm of $\mathbf{R}$ vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

`GEN gsupnorm(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

`void gsupnorm_aux(GEN x, GEN *m, GEN *m2, long prec)` is the low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of x is the min of `m` and the square root of `m2`; one or both of `m`, `m2` may be `NULL`, in which case it must be omitted. You may initially set `m` and `m2` to non-`NULL` values, in which case, the above procedure yields the max of (the initial) `m`, the square root of (the initial) `m2`, and the sup norm of x.

The strange interface is due to the fact that $|z|^2$ is easier to compute than $|z|$ for a `t_QUAD` or `t_COMPLEX` z: `m2` is the max of those $|z|^2$, and `m` is the max of the other $|z|$.

9.14 Substitution and evaluation.

`GEN gsubst(GEN x, long v, GEN y)` substitutes the object `y` into `x` for the variable number `v`.

`GEN poleval(GEN q, GEN x)` evaluates the `t_POL` or `t_RFRAC` q at x. For convenience, a `t_VEC` or `t_COL` is also recognized as the `t_POL` `gtovecrev(q)`.

`GEN RgX_cxeval(GEN T, GEN x, GEN xi)` evaluate the `t_POL` T at x via Horner's scheme. If xi is not `NULL` it must be equal to $1/x$ and we evaluate $x^{\deg T} T(1/x)$ instead. This is useful when $|x| > 1$ is a `t_REAL` or an inexact `t_COMPLEX` and T has "balanced" coefficients, since the evaluation becomes numerically stable.

`GEN RgXY_cxevalx(GEN T, GEN x, GEN xi)` Apply `RgX_cxeval` to all the polynomials coefficients of T.

`GEN RgX_RgM_eval(GEN q, GEN x)` evaluates the `t_POL` q at the square matrix x.

`GEN RgX_RgMV_eval(GEN f, GEN V)` returns the evaluation `f(x)`, assuming that `V` was computed by `FpXQ_powers(x,`n`)` for some $n > 1$.

`GEN qfeval(GEN q, GEN x)` evaluates the quadratic form q (symmetric matrix) at x (column vector of compatible dimensions).

`GEN qfevalb(GEN q, GEN x, GEN y)` evaluates the polar bilinear form attached to the quadratic form q (symmetric matrix) at x, y (column vectors of compatible dimensions).

`GEN hqfeval(GEN q, GEN x)` evaluates the Hermitian form q (a Hermitian complex matrix) at x.

`GEN qf_RgM_apply(GEN q, GEN M)` q is a symmetric $n \times n$ matrix, M an $n \times k$ matrix, return $M'qM$.

`GEN qf_ZM_apply(GEN q, GEN M)` as above assuming that both q and M have integer entries.

Chapter 10:
Miscellaneous mathematical functions

10.1 Fractions.

`GEN absfrac(GEN x)` returns the absolute value of the `t_FRAC` x.

`GEN absfrac_shallow(GEN x)` x being a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(`x`)` otherwise.

`GEN sqrfrac(GEN x)` returns the square of the `t_FRAC` x.

10.2 Binomials.

`GEN binomial(GEN x, long k)`

`GEN binomialuu(ulong n, ulong k)`

`GEN vecbinomial(long n)`, which returns a vector v with $n+1$ `t_INT` components such that $v[k+1] =$ `binomial`(n,k) for k from 0 up to n.

10.3 Real numbers.

`GEN R_abs(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns $|x|$.

`GEN R_abs_shallow(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(`x`)` otherwise.

`GEN modRr_safe(GEN x, GEN y)` let x be a `t_INT`, a `t_REAL` or `t_FRAC` and let y be a `t_REAL`. Return $x\%y$ unless the input accuracy is unsufficient to compute the floor or x/y in which case we return `NULL`.

`GEN modRr_i(GEN x, GEN y, GEN iy)` let x be a `t_INT`, a `t_REAL` or `t_FRAC` and let y be a `t_REAL` and $iy =$ `invr`(y). Return $x\%y$ unless the input accuracy is unsufficient to compute the floor or x/y in which case we return `NULL`.

10.4 Complex numbers.

`GEN gimag(GEN x)` returns a copy of the imaginary part of `x`.

`GEN greal(GEN x)` returns a copy of the real part of `x`. If `x` is a `t_QUAD`, returns the coefficient of 1 in the "canonical" integral basis $(1, \omega)$.

`GEN gconj(GEN x)` returns $\texttt{greal}(x) - 2\texttt{gimag}(x)$, which is the ordinary complex conjugate except for a real `t_QUAD`.

`GEN imag_i(GEN x)`, shallow variant of `gimag`.

`GEN real_i(GEN x)`, shallow variant of `greal`.

`GEN conj_i(GEN x)`, shallow variant of `gconj`.

`GEN mulreal(GEN x, GEN y)` returns the real part of xy; x, y have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`. See also `RgM_mulreal`.

`GEN mulimag(GEN x, GEN y)` returns the imaginary part of xy; x, y have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`.

`GEN cxnorm(GEN x)` norm of the `t_COMPLEX` x (modulus squared).

`GEN cxexpm1(GEN x)` returns $\exp(x) - 1$, for a `t_COMPLEX` x.

`int cx_approx_equal(GEN a, GEN b)` test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those) a and b are approximately equal. This returns 1 if and only if the division by $a - b$ would produce a division by 0 (which is a less stringent test than testing whether $a - b$ evaluates to 0).

`int cx_approx0(GEN a, GEN b)` test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those) a is approximately 0, where b is a reference point. A non-0 `t_REAL` component x is approximately 0 if

$$\texttt{exponent}(b) - \texttt{exponent}(x) > \texttt{bit\_prec}(\texttt{x}) \ .$$

10.5 Low accuracy input or output.

`double dbllog2r(GEN x)` assuming that `x` is a nonzero `t_REAL`, returns an approximation to `log2(|x|)`.

`double dbllog2(GEN x)` assuming that `x` is a complex number of type `t_INT`, `t_FRAC`, `t_REAL` or a `t_COMPLEX` (with real components) returns an approximation to `log2(|x|)`. Returns 1./0. if x is 0.

`double dbllambertW0(double y)` given $y > e^{-1}$ return a rough approximation x to $W_0(y)$, where W_0 is the principal branch of the Lambert function (relative error less than 1%). We thus have $xe^x \approx y$.

`double dbllambertW_1(double y)` given $0 > y > e^{-1}$ return a rough approximation x to $W_{-1}(y)$ (relative error less than 1%). We thus have $xe^x \approx y$.

Finally the following 4 functions provide low-accuracy evaluations at the complex number $z = s + it$:

`double dblcnorm(double s, double t)` returns $|z|^2 = s^2 + t^2$.

`double dblcabs(double s, double t)` returns $|z| = \sqrt{s^2 + t^2}$.

`double dblcarg(double s, double t)` returns $\arg z$, with values in $]-\pi,\pi]$. Assumes $z \neq 0$.

`void dblclog(double s, double t, double *a, double *b)` sets $a + ib = \mathrm{Log}(z)$, where Log is the principal branch of the natural logarithm, with imaginary part in $]-\pi,\pi]$. Assumes $z \neq 0$.

10.6 Quadratic numbers and binary quadratic forms.

`GEN quad_disc(GEN x)` returns the discriminant of the `t_QUAD` x. Not stack-clean but suitable for `gc_upto`.

`GEN quadnorm(GEN x)` norm of the `t_QUAD` x.

`GEN qfb_disc(GEN x)` returns the discriminant of the `t_QFB x`.

`GEN qfb_disc3(GEN x, GEN y, GEN z)` returns $y^2 - 4xz$ assuming all inputs are `t_INT`s. Not stack-clean.

`GEN qfb_ZM_apply(GEN q, GEN g)` returns $q \circ g$.

`GEN qfbforms(GEN D)` given a discriminant $D < 0$, return the list of reduced forms of discriminant D as `t_VECSMALL` with 3 components. The primitive forms in the list enumerate the class group of the quadratic order of discriminant D; if D is fundamental, all returned forms are automatically primitive.

10.7 Polynomials.

`GEN truecoef(GEN x, long n)` returns `polcoef(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that *copies* its return value to ease garbage collection. Use `polcoef_i` for a fast shallow variant.

`GEN polcoef_i(GEN x, long n, long v)` shallow function. Rewrite x as a Laurent polynomial in the variable v and returns its coefficient of degree n (`gen_0` if this falls outside the coefficient array). Allow `t_POL`, `t_SER`, `t_RFRAC` and scalars.

`long degree(GEN x)` returns `poldegree(x, -1)`, the degree of `x` with respect to its main variable, with the usual meaning if the leading coefficient of x is nonzero. If the sign of x is 0, this function always returns -1. Otherwise, we return the index of the leading coefficient of x, i.e. the coefficient of largest index stored in x. For instance the "degrees" of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and -1 respectively.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL x` with respect to its main variable, *if* its leading coefficient is nonzero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`GEN characteristic(GEN x)` returns the characteristic of the base ring over which the polynomial is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN residual_characteristic(GEN x)` returns a kind of "residual characteristic" of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli `t_INTMOD`s occurring

in the structure, as well as primes p arising from `t_PADIC`s or `t_FFELT`s. The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN resultant(GEN x, GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in $\mathbf{Q}[X]$) or Sylvester's matrix (inexact inputs).

`GEN resultant2(GEN x, GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Computes the determinant of Sylvester's matrix.

`GEN cleanroots(GEN x, long prec)` returns the complex roots of the complex polynomial x (with coefficients `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` of the above). The roots are returned as `t_REAL` or `t_COMPLEX` of `t_REAL`s of precision `prec` (guaranteeing a nonzero imaginary part). See `QX_complex_roots`.

`double fujiwara_bound(GEN x)` return a quick upper bound for the logarithm in base 2 of the modulus of the largest complex roots of the polynomial x (complex coefficients).

`double fujiwara_bound_real(GEN x, long sign)` return a quick upper bound for the logarithm in base 2 of the absolute value of the largest real root of sign *sign* (1 or -1), for the polynomial x (real coefficients).

`GEN polmod_to_embed(GEN x, long prec)` return the vector of complex embeddings of the `t_POLMOD` x (with complex coefficients). Shallow function, simple complex variant of `conjvec`.

`GEN pollegendre_reduced(long n, long v)` let $P_n(t) \in \mathbf{Q}[t]$ be the n-th Legendre polynomial in variable v. Return $p \in \mathbf{Z}[t]$ such that $2^n P_n(t) = p(t^2)$ (n even) or $tp(t^2)$ (n odd).

10.7.1 Obsolete functions.

`long sturmpart(GEN x, GEN a, GEN b)`. Please use `RgX_sturmpart` instead, and preferably `ZX_sturmpart` with exact rational endpoints.

10.8 Power series.

`GEN sertoser(GEN x, long prec)` return the `t_SER` x truncated or extended (with zeros) to `prec` terms. Shallow function, assume that `prec` ≥ 0.

`GEN derivser(GEN x)` returns the derivative of the `t_SER` `x` with respect to its main variable.

`GEN integser(GEN x)` returns the primitive of the `t_SER` `x` with respect to its main variable.

`GEN truecoef(GEN x, long n)` returns `polcoef(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that *copies* its return value to simplify garbage collection. Use `polcoef_i` for a fast shallow variant.

`GEN ser_unscale(GEN P, GEN h)` return $P(hx)$, not stack-clean.

`GEN ser_normalize(GEN x)` divide x by its "leading term" so that the series is either 0 or equal to $t^v(1 + O(t))$. Shallow function if the "leading term" is 1.

`int ser_isexactzero(GEN x)` return 1 if x is a zero series, all of whose known coefficients are exact zeroes; this implies that `sign`$(x) = 0$ and `lg`$(x) \leq 3$.

`GEN ser_inv(GEN x)` return the inverse of the `t_SER` x using Newton iteration.

`GEN psi1series(long n, long v, long prec)` creates the `t_SER` $\psi(1 + x + O(x^n))$ in variable v.

10.9 Functions to handle t_FFELT.

These functions define the public interface of the t_FFELT type to use in generic functions. However, in specific functions, it is better to use the functions class FpXQ and/or Flxq as appropriate.

GEN FF_p(GEN a) returns the characteristic of the definition field of the t_FFELT element a.

long FF_f(GEN a) returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus p^f.

GEN FF_p_i(GEN a) shallow version of FF_p.

GEN FF_q(GEN a) returns the cardinality of the definition field of the t_FFELT element a.

GEN FF_mod(GEN a) returns the polynomial (with reduced t_INT coefficients) defining the finite field, in the variable used to display a.

long FF_var(GEN a) returns the variable used to display a.

GEN FF_gen(GEN a) returns the standard generator of the definition field of the t_FFELT element a, see ffgen, that is $x \pmod T$ where T is the polynomial over the prime field that define the finite field.

GEN FF_to_FpXQ(GEN a) converts the t_FFELT a to a polynomial P with reduced t_INT coefficients such that $a = P(g)$ where g is the generator of the finite field returned by ffgen, in the variable used to display g.

GEN FF_to_FpXQ_i(GEN a) shallow version of FF_to_FpXQ.

GEN FF_to_F2xq(GEN a) converts the t_FFELT a to a F2x P such that $a = P(g)$ where g is the generator of the finite field returned by ffgen, in the variable used to display g. This only work if the characteristic is 2.

GEN FF_to_F2xq_i(GEN a) shallow version of FF_to_F2xq.

GEN FF_to_Flxq(GEN a) converts the t_FFELT a to a Flx P such that $a = P(g)$ where g is the generator of the finite field returned by ffgen, in the variable used to display g. This only work if the characteristic is small enough.

GEN FF_to_Flxq_i(GEN a) shallow version of FF_to_Flxq.

GEN p_to_FF(GEN p, long v) returns a t_FFELT equal to 1 in the finite field $\mathbf{Z}/p\mathbf{Z}$. Useful for generic code that wants to handle (inefficiently) $\mathbf{Z}/p\mathbf{Z}$ as if it were not a prime field.

GEN Tp_to_FF(GEN T, GEN p) returns a t_FFELT equal to 1 in the finite field $\mathbf{F}_p/(T)$, where T is a ZX, assumed to be irreducible modulo p, or NULL in which case the routine acts as p_to_FF(p,0). No checks.

GEN Fq_to_FF(GEN x, GEN ff) returns a t_FFELT equal to x in the finite field defined by the t_FFELT ff, where x is an Fq (either a t_INT or a ZX: a t_POL with t_INT coefficients). No checks.

GEN FqC_to_FFC(GEN x, GEN ff) given an FqC x, return the t_COL with t_FFELT coefficients obtained by applying Fq_to_FF coefficientwise. No checks.

GEN FqV_to_FFV(GEN x, GEN ff) given an FqV x, return the t_VEC with t_FFELT coefficients obtained by applying Fq_to_FF coefficientwise. No checks.

GEN FqM_to_FFM(GEN x, GEN ff) given an FqM x, return the t_MAT with t_FFELT coefficients obtained by applying Fq_to_FF coefficientwise. No checks.

`GEN FqX_to_FFX(GEN x, GEN ff)` given an `FqX` x, return the polynomial with `t_FFELT` coefficients obtained by applying `Fq_to_FF` coefficientwise. No checks, and no normalization if the leading coefficient maps to 0.

`GEN FF_1(GEN a)` returns the unity in the definition field of the `t_FFELT` element `a`.

`GEN FF_zero(GEN a)` returns the zero element of the definition field of the `t_FFELT` element `a`.

`int FF_equal0(GEN a)` returns 1 if the `t_FFELT` `a` is equal to 0 else returns 0.

`int FF_equal1(GEN a)` returns 1 if the `t_FFELT` `a` is equal to 1 else returns 0.

`int FF_equalm1(GEN a)` returns -1 if the `t_FFELT` `a` is equal to 1 else returns 0.

`int FF_equal(GEN a, GEN b)` return 1 if the `t_FFELT` `a` and `b` have the same definition field and are equal, else 0.

`int FF_samefield(GEN a, GEN b)` return 1 if the `t_FFELT` `a` and `b` have the same definition field, else 0.

`int Rg_is_FF(GEN c, GEN *ff)` to be called successively on many objects, setting `*ff = NULL` (unset) initially. Returns 1 as long as c is a `t_FFELT` defined over the same field as `*ff` (setting `*ff = c` if unset), and 0 otherwise.

`int RgC_is_FFC(GEN x, GEN *ff)` apply `Rg_is_FF` successively to all components of the `t_VEC` or `t_COL` x. Return 0 if one call fails, and 1 otherwise.

`int RgM_is_FFM(GEN x, GEN *ff)` apply `Rg_is_FF` to all components of the `t_MAT`. Return 0 if one call fails, and 1 otherwise.

`GEN FF_add(GEN a, GEN b)` returns $a + b$ where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_Z_add(GEN a, GEN x)` returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_Q_add(GEN a, GEN x)` returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_RFRAC`, the computation being performed in the definition field of `a`.

`GEN FF_sub(GEN a, GEN b)` returns $a - b$ where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_mul(GEN a, GEN b)` returns ab where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_Z_mul(GEN a, GEN b)` returns ab, where `a` is a `t_FFELT`, and `b` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_div(GEN a, GEN b)` returns a/b where `a` and `b` are `t_FFELT` having the same definition field.

`GEN FF_neg(GEN a)` returns $-a$ where `a` is a `t_FFELT`.

`GEN FF_neg_i(GEN a)` shallow function returning $-a$ where `a` is a `t_FFELT`.

`GEN FF_inv(GEN a)` returns a^{-1} where `a` is a `t_FFELT`.

`GEN FF_sqr(GEN a)` returns a^2 where `a` is a `t_FFELT`.

`GEN FF_mul2n(GEN a, long n)` returns $a2^n$ where `a` is a `t_FFELT`.

`GEN FF_pow(GEN a, GEN n)` returns a^n where `a` is a `t_FFELT` and `n` is a `t_INT`.

`GEN FF_Frobenius(GEN a, GEN n)` returns x^{p^n} where `x` is the standard generator of the definition field of the `t_FFELT` element `a`, `t_FFELT`, `n` is a `t_INT`, and p is the characteristic of the definition field of a.

`GEN FF_Z_Z_muldiv(GEN a, GEN x, GEN y)` returns ay/z, where `a` is a `t_FFELT`, and `x` and `y` are `t_INT`, the computation being performed in the definition field of `a`.

`GEN Z_FF_div(GEN x, GEN a)` return x/a where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

`GEN FF_norm(GEN a)` returns the norm of the `t_FFELT a` with respect to its definition field.

`GEN FF_trace(GEN a)` returns the trace of the `t_FFELT a` with respect to its definition field.

`GEN FF_conjvec(GEN a)` returns the vector of conjugates $[a, a^p, a^{p^2}, \ldots, a^{p^{n-1}}]$ where the `t_FFELT a` belong to a field with p^n elements.

`GEN FF_charpoly(GEN a)` returns the characteristic polynomial) of the `t_FFELT a` with respect to its definition field.

`GEN FF_minpoly(GEN a)` returns the minimal polynomial of the `t_FFELT a`.

`GEN FF_sqrt(GEN a)` returns an `t_FFELT` b such that $a = b^2$ if it exist, where `a` is a `t_FFELT`.

`long FF_issquareall(GEN x, GEN *pt)` returns 1 if `x` is a square, and 0 otherwise. If `x` is indeed a square, set `pt` to its square root.

`long FF_issquare(GEN x)` returns 1 if `x` is a square and 0 otherwise.

`long FF_ispower(GEN x, GEN K, GEN *pt)` Given K a positive integer, returns 1 if `x` is a K-th power, and 0 otherwise. If `x` is indeed a K-th power, set `pt` to its K-th root.

`GEN FF_sqrtn(GEN a, GEN n, GEN *zn)` returns an `n`-th root of `a` if it exist. If `zn` is non-`NULL` set it to a primitive `n`-th root of the unity.

`GEN FF_log(GEN a, GEN g, GEN o)` the `t_FFELT g` being a generator for the definition field of the `t_FFELT a`, returns a `t_INT` e such that $a^e = g$. If e does not exists, the result is currently undefined. If `o` is not `NULL` it is assumed to be a factorization of the multiplicative order of `g` (as set by `FF_primroot`)

`GEN FF_order(GEN a, GEN o)` returns the order of the `t_FFELT a`. If `o` is non-`NULL`, it is assumed that `o` is a multiple of the order of `a`.

`GEN FF_primroot(GEN a, GEN *o)` returns a generator of the multiplicative group of the definition field of the `t_FFELT a`. If `o` is not `NULL`, set it to the factorization of the order of the primitive root (to speed up `FF_log`).

`GEN FF_map(GEN m, GEN a)` returns $A(m)$ where `A=a.pol` assuming a and m belongs to fields having the same characteristic.

10.9.1 FFX.

The functions in this sections take polynomial arguments and a `t_FFELT` a. The coefficients of the polynomials must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_add(GEN P, GEN Q, GEN a)` returns the sum of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_mul(GEN P, GEN Q, GEN a)` returns the product of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_sqr(GEN P, GEN a)` returns the square of the polynomial `P` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_rem(GEN P, GEN Q, GEN a)` returns the remainder of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_gcd(GEN P, GEN Q, GEN a)` returns the GCD of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_extgcd(GEN P, GEN Q, GEN a, GEN *U, GEN *V)` returns the GCD of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` `a` and sets `*U`, `*V` to the Bezout coefficients such that $*\texttt{U}\texttt{P} + *\texttt{V}\texttt{Q} = d$. If `*U` is set to `NULL`, it is not computed which is a bit faster.

`GEN FFX_halfgcd(GEN x, GEN y, GEN a)` returns a two-by-two matrix M with determinant ± 1 such that the image (A, B) of (x, y) by M has the property that $\deg A \geq \frac{\deg x}{2} > \deg B$.

`GEN FFX_halfgcd_all(GEN x, GEN y, GEN a, GEN *ptA, GEN *ptB)` as `FFX_halfgcd`, in addition, if `ptA` (resp. `ptB`) is not `NULL`, `*ptA` (resp. `*ptB`) is set to `A` (resp. `B`).

`GEN FFX_resultant(GEN P, GEN Q, GEN a)` returns the resultant of the polynomials `P` and `Q` where `P` and `Q` are defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_disc(GEN P, GEN a)` returns the discriminant of the polynomial `P` where `P` is defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_ispower(GEN P, ulong k, GEN a, GEN *py)` return 1 if the `FFX` P is a k-th power, 0 otherwise, where `P` is defined over the definition field of the `t_FFELT` `a`. If `py` is not `NULL`, set it to g such that $g^k = f$.

`GEN FFX_factor(GEN f, GEN a)` returns the factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` `a`. The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_factor_squarefree(GEN f, GEN a)` returns the squarefree factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` `a`. This is a vector $[u_1, \ldots, u_k]$ of pairwise coprime `FFX` such that $u_k \neq 1$ and $f = \prod u_i^i$.

`GEN FFX_ddf(GEN f, GEN a)` assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value `v` is a `t_VEC` with two components: `F=v[1]` is a vector of (`FFX`) factors, and `E=v[2]` is a `t_VECSMALL`, such that f is equal to the product of the `F[i]` and each `F[i]` is a product of irreducible factors of degree `E[i]`.

`GEN FFX_degfact(GEN f, GEN a)`, as `FFX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`).

GEN FFX_roots(GEN f, GEN a) returns the roots (t_FFELT) of the univariate polynomial f over the definition field of the t_FFELT a. The coefficients of f must be of type t_INT, t_INTMOD or t_FFELT and compatible with a.

long FFX_nbroots(GEN f, GEN a) returns the number of roots of the univariate polynomial f over the definition field of the t_FFELT a. The coefficients of f must be of type t_INT, t_INTMOD or t_FFELT and compatible with a.

GEN FFX_preimagerel(GEN F, GEN x, GEN a) returns $P\%F$ where P=x.pol assuming a and x belongs to fields having the same characteristic, and that the coefficients of F belong to the definition field of a.

GEN FFX_preimage(GEN F, GEN x, GEN a) as FFX_preimagerel but return NULL if the remainder is of degree greater or equal to 1, the constant coefficient otherwise.

GEN FFV_roots_to_pol(GEN V, GEN ff, long v) V being a vector over the finite field given by ff (t_FFELT), returns the monic t_POL $\prod_i$(pol_x[v] − V[i]).

10.9.2 FFM.

GEN FFM_FFC_gauss(GEN M, GEN C, GEN ff) given a matrix M (t_MAT) and a column vector C (t_COL) over the finite field given by ff (t_FFELT) such that M is invertible, return the unique column vector X such that $MX = C$.

GEN FFM_FFC_invimage(GEN M, GEN C, GEN ff) given a matrix M (t_MAT) and a column vector C (t_COL) over the finite field given by ff (t_FFELT), return a column vector X such that $MX = C$, or NULL if no such vector exists.

GEN FFM_FFC_mul(GEN M, GEN C, GEN ff) returns the product of the matrix M (t_MAT) and the column vector C (t_COL) over the finite field given by ff (t_FFELT).

GEN FFM_deplin(GEN M, GEN ff) returns a nonzero vector (t_COL) in the kernel of the matrix M over the finite field given by ff, or NULL if no such vector exists.

GEN FFM_det(GEN M, GEN ff) returns the determinant of the matrix M over the finite field given by ff.

GEN FFM_gauss(GEN M, GEN N, GEN ff) given two matrices M and N (t_MAT) over the finite field given by ff (t_FFELT) such that M is invertible, return the unique matrix X such that $MX = N$.

GEN FFM_image(GEN M, GEN ff) returns a matrix whose columns span the image of the matrix M over the finite field given by ff.

GEN FFM_indexrank(GEN M, GEN ff) given a matrix M of rank r over the finite field given by ff, returns a vector with two t_VECSMALL components y and z containing r row and column indices, respectively, such that the $r \times r$-matrix formed by the M[i,j] for i in y and j in z is invertible.

GEN FFM_inv(GEN M, GEN ff) returns the inverse of the square matrix M over the finite field given by ff, or NULL if M is not invertible.

GEN FFM_invimage(GEN M, GEN N, GEN ff) given two matrices M and N (t_MAT) over the finite field given by ff (t_FFELT), return a matrix X such that $MX = N$, or NULL if no such matrix exists.

GEN FFM_ker(GEN M, GEN ff) returns the kernel of the t_MAT M over the finite field given by the t_FFELT ff.

`GEN FFM_mul(GEN M, GEN N, GEN ff)` returns the product of the matrices `M` and `N` (`t_MAT`) over the finite field given by `ff` (`t_FFELT`).

`long FFM_rank(GEN M, GEN ff)` returns the rank of the matrix `M` over the finite field given by `ff`.

`GEN FFM_suppl(GEN M, GEN ff)` given a matrix `M` over the finite field given by `ff` whose columns are linearly independent, returns a square invertible matrix whose first columns are those of `M`.

10.9.3 FFXQ.

`GEN FFXQ_mul(GEN P, GEN Q, GEN T, GEN a)` returns the product of the polynomials `P` and `Q` modulo the polynomial `T`, where `P`, `Q` and `T` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_sqr(GEN P, GEN T, GEN a)` returns the square of the polynomial `P` modulo the polynomial `T`, where `P` and `T` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_inv(GEN P, GEN Q, GEN a)` returns the inverse of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_minpoly(GEN Pf, GEN Qf, GEN a)` returns the minimal polynomial of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT a`.

10.10 Transcendental functions.

The following two functions are only useful when interacting with `gp`, to manipulate its internal default precision (expressed as a number of decimal digits, not in bits as used everywhere else):

`long getrealprecision(void)` returns `realprecision`.

`long setrealprecision(long n, long *prec)` sets the new `realprecision` to n, which is returned. As a side effect, set `prec` to the corresponding bit precision `ndec2prec(n)`.

10.10.1 Transcendental functions with `t_REAL` arguments.

In the following routines, x is assumed to be a `t_REAL` and the result is a `t_REAL` (sometimes a `t_COMPLEX` with `t_REAL` components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix `mp` even though `t_INT` inputs are forbidden:

`GEN sqrtr(GEN x)` returns the square root of x.

`GEN cbrtr(GEN x)` returns the real cube root of x.

`GEN sqrtnr(GEN x, long n)` returns the n-th root of x, assuming $n \geq 1$ and $x \geq 0$.

`GEN sqrtnr_abs(GEN x, long n)` returns the n-th root of $|x|$, assuming $n \geq 1$ and $x \neq 0$.

`GEN mpcos(GEN x)` returns $\cos(x)$.

`GEN mpsin(GEN x)` returns $\sin(x)$.

`GEN mpsinc(GEN x)` returns $\sin(x)/x$.

`GEN mptan(GEN x)` returns $\tan(x)$, assumes $\cos(x) \neq 0$.

`GEN mpcotan(GEN x)` returns $\mathrm{cotan}(x)$, assumes $\sin(x) \neq 0$.

`GEN mpacos(GEN x)` returns $\mathrm{acos}(x)$, assumes $|x| \leq 1$.

`GEN mpasin(GEN x)` returns $\mathrm{asin}(x)$, assumes $|x| \leq 1$.

`GEN mpatan(GEN x)` returns $\mathrm{atan}(x)$.

`GEN mpcosh(GEN x)` returns $\cosh(x)$.

`GEN mpsinh(GEN x)` returns $\sinh(x)$.

`GEN mptanh(GEN x)` returns $\tanh(x)$.

`GEN mpcotanh(GEN x)` returns $\mathrm{cotanh}(x)$, $x \neq 0$.

`GEN mpacosh(GEN x)` returns $\mathrm{acosh}(x)$, assumes $|x| \geq 1$.

`GEN mpasinh(GEN x)` returns $\mathrm{asinh}(x)$.

`GEN mpatanh(GEN x)` returns $\mathrm{atanh}(x)$, assumes $|x| < 1$.

`GEN mplog(GEN x)` returns $\log(x)$. We must have $x > 0$ since the result must be a `t_REAL`. Use `glog` for the general case, where you want such computations as $\log(-1) = I$.

`GEN mplog1p(GEN x)` returns $\log(1 + x)$, computed in a way that is also accurate near $x = 0$.

`GEN mpexp(GEN x)` returns $\exp(x)$.

`GEN mpexpm1(GEN x)` returns $\exp(x) - 1$, but is more accurate than `subrs(mpexp(x), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`void mpsincosm1(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x) - 1$ respectively, where x is a `t_REAL`; the latter is more accurate than `subrs(mpcos(y), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`GEN mpveceint1(GEN C, GEN eC, long n)` as `veceint1`; assumes that $C > 0$ is a `t_REAL` and that `eC` is `NULL` or `mpexp(C)`.

`GEN mpeint1(GEN x, GEN expx)` returns `eint1`(x), for a `t_REAL` $x \neq 0$, assuming that `expx` is `mpexp`(x).

A few variants on the Lambert function: they actually work when `gtofp` can map all `GEN` arguments to a `t_REAL`.

`GEN mplambertW(GEN y)` solution $x = W_0(y)$ of the implicit equation $x \exp(x) = y$, for $y > -1/e$ a `t_REAL`.

`GEN mplambertx_logx(GEN a, GEN b, long bit)` solve $x - a\log(x) = b$ with $a > 0$ and $b \geq a(1 - \log(a))$.

`GEN mplambertX(GEN y, long bit)` as `mplambertx_logx` in the special case $a = 1$, $b = \log(y)$. In other words, solve $e^x/x = y$ with $y \geq e$.

`GEN mplambertxlogx_x(GEN a, GEN b, long bit)` solve $x\log(x) - ax = b$; if $b < 0$, assume $a \geq 1 + \log|b|$.

Useful low-level functions which *disregard* the sign of x:

`GEN sqrtr_abs(GEN x)` returns $\sqrt{|x|}$ assuming $x \neq 0$.

`GEN cbrtr_abs(GEN x)` returns $|x|^{1/3}$ assuming $x \neq 0$.

`GEN exp1r_abs(GEN x)` returns $\exp(|x|) - 1$, assuming $x \neq 0$.

`GEN logr_abs(GEN x)` returns $\log(|x|)$, assuming $x \neq 0$.

10.10.2 Other complex transcendental functions.

`GEN atanhuu(ulong u, ulong v, long prec)` computes $\operatorname{atanh}(u/v)$ using binary splitting, assuming $0 < u < v$. Not stack-clean but suitable for `gc_upto`. The complexity is $O(\texttt{prec}/\log(v/u))$ so becomes impractical when v/u is too close to 1.

`GEN atanhui(ulong u, GEN v, long prec)` computes $\operatorname{atanh}(u/v)$ using binary splitting, assuming $0 < u < v$. Not stack-clean but suitable for `gc_upto`.

`GEN szeta(long s, long prec)` returns the value of Riemann's zeta function at the (possibly negative) integer $s \neq 1$, in relative accuracy `prec`.

`GEN veczeta(GEN a, GEN b, long N, long prec)` returns in a vector all the $\zeta(aj + b)$, where $j = 0, 1, \ldots, N - 1$, where a and b are real numbers (of arbitrary type, although `t_INT` is treated more efficiently). Assumes that $N \geq 1$, $b > 1$ and $a(N - 1) + b > 1$ (in case $a < 0$); the silly case $a = 0$ is wasteful but allowed.

`GEN ggamma1m1(GEN x, long prec)` return $\Gamma(1 + x) - 1$ assuming $|x| < 1$. Guard against cancellation when x is small.

A few variants on sin and cos:

`void mpsincos(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x)$ respectively, where x is a `t_REAL`

`void mpsinhcosh(GEN x, GEN *s, GEN *c)` sets s and c to $\sinh(x)$ and $\cosh(x)$ respectively, where x is a `t_REAL`

`GEN expIr(GEN x)` returns $\exp(ix)$, where x is a `t_REAL`. The return type is `t_COMPLEX` unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

`GEN expIPiQ(GEN x, long prec)` returns $\exp(i\pi x)$, where x is a rational number (`t_INT` or `t_FRAC`). Contrary to general transcendental functions, `expIPiQ` returns a result with exact components if possible. E.g, it returns `gen_1` or `gen_m1` when x is integral.

`GEN expIPiR(GEN x, long prec)` returns $\exp(i\pi x)$, where x is a real number (`t_INT`, `t_FRAC` or `t_REAL`). Calls `expIPiQ` for rational arguments and returns a result with exact components if possible.

`GEN expIPiC(GEN z, long prec)` return $\exp(i\pi x)$, where x is a complex number (`t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`). Calls `expIPiR` for real arguments and returns a result with exact components if possible.

`GEN expIxy(GEN x, GEN y, long prec)` returns $\exp(ixy)$. Efficient when x is real and y pure imaginary.

`GEN pow2Pis(GEN s, long prec)` returns $(2\pi)^s$. The intent of this function and the next ones is to be accurate even if s has a huge imaginary part: π is computed at an accuracy taking into account the cancellation induced by argument reduction when computing the sine or cosine of $\Im s \log 2\pi$.

`GEN powPis(GEN s, long prec)` returns π^s, as `pow2Pis`.

`long powcx_prec(long e, GEN s, long prec)` if $e \approx \log_2 |x|$ return the precision at which $\log(x)$ must be computed to evaluate x^s reliably (taking into account argument reduction).

`GEN powcx(GEN x, GEN logx, GEN s, long prec)` assuming s is a `t_COMPLEX` and `logx` is $\log(x)$ computed to accuracy `powcx_prec`, return x^s.

`void gsincos(GEN x, GEN *s, GEN *c, long prec)` general case.

`GEN rootsof1_cx(GEN d, long prec)` return $e(1/d)$ at precision `prec`, $e(x) = \exp(2i\pi x)$.

`GEN rootsof1u_cx(ulong d, long prec)` return $e(1/d)$ at precision `prec`.

`GEN rootsof1q_cx(long a, long b, long prec)` return $e(a/b)$ at precision `prec`.

`GEN rootsof1powinit(long a, long b, long prec)` precompute b-th roots of 1 for `rootsof1pow`, i.e. to later compute $e(ac/b)$ for varying c.

`GEN rootsof1pow(GEN T, long c)` given $T =$ `rootsof1powinit`$(a, b,$ `prec`$)$, return $e(ac/b)$.

`void affgc(GEN x, GEN res)` assume `res` `res` was allocated using `cgetc`, and that x is a complex number (`t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` with real components). Assign x to `res`. Generalizes `affgr`.

`GEN affc_fixlg(GEN x, GEN res)` assume `res` was allocated using `cgetc`, and that x is either a `t_REAL` or a `t_COMPLEX` with `t_REAL` components. Assign x to `res`, first shortening the components of `res` if needed (in a stack-safe way). Further convert `res` to a `t_REAL` if x is a `t_REAL`. Generalizes `affrr_fixlg`.

`GEN trans_eval(const char *fun, GEN (*f) (GEN, long), GEN x, long prec)` evaluate the transcendental function f (named `"fun"` at the argument x and precision `prec`. This is a quick way to implement a transcendental function to be made available under GP, starting from a C function handling only `t_REAL` and `t_COMPLEX` arguments. This routine first converts x to a suitable type:

- `t_INT`/`t_FRAC` to `t_REAL` of precision `prec`, `t_QUAD` to `t_REAL` or `t_COMPLEX` of precision `prec`.

- `t_POLMOD` to a `t_COL` of complex embeddings (as in `conjvec`)

Then evaluates the function at `t_VEC`, `t_COL`, `t_MAT` arguments coefficientwise.

`GEN trans_evalgen(const char *fun, void *E, GEN (*f)(void*, GEN, long), GEN x, long prec)`, general variant evaluating $f(E, x, prec)$, where the function prototype allows to wrap an arbitrary context given by the argument E.

10.10.3 Modular functions.

`GEN cxredsl2(GEN z, GEN *g)` given t a `t_COMPLEX` belonging to the upper half-plane, find $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma \cdot z$ belongs to the standard fundamental domain and set `*g` to γ.

`GEN cxredsl2_i(GEN z, GEN *g, GEN *czd)` as `cxredsl2`; also sets `*czd` to $cz+d$, if $\gamma = [a, b; c, d]$.

`GEN cxEk(GEN tau, long k, long prec)` returns $E_k(\tau)$ by direct evaluation of $1 + 2/\zeta(1-k)\sum_n n^{k-1}q^n/(1-q^n)$, $q = e(\tau)$. Assume that $\Im\tau > 0$ and k even. Very slow unless τ is already reduced modulo $\mathrm{SL}_2(\mathbf{Z})$. Not stack-clean but suitable for `gc_upto`.

10.10.4 Transcendental functions with t_PADIC arguments.

The argument x is assumed to be a t_PADIC.

GEN Qp_exp(GEN x) shortcut for gexp(x, /*ignored*/prec)

long Qp_exp_prec(GEN x) number of terms to sum in the $\exp(x)$ series to reach the same p-adic accuracy as $x \neq 0$. If $n = p - 1$, $e = v_p(x)$ and $b =$ precp(x), this is the ceiling of $nb/(ne - 1)$. Return -1 if the series does not converge ($ne \leq 1$).

GEN Qp_gamma(GEN x) shortcut for ggamma(x, /*ignored*/prec)

GEN Qp_zeta(GEN x) shortcut for gzeta(x, /*ignored*/prec); assume that $x \neq 1$.

GEN Qp_zetahurwitz(GEN x, GEN y, long k) shortcut for zetahurwitz(x, y, -k, /*ignored*/prec).

GEN Qp_psi(GEN x, long der) shortcut for gpsi_der(x, der, /*ignored*/prec).

GEN Qp_log(GEN x) shortcut for glog(x, /*ignored*/prec).

GEN Qp_sqrt(GEN x) shortcut for gsqrt(x, /*ignored*/prec) Return NULL if x is not a square.

GEN Qp_sqrtn(GEN x, GEN n, GEN *z) shortcut for gsqrtn(x, n, z, /*ignored*/prec). Return NULL if x is not an n-th power.

GEN Qp_agm2_sequence(GEN a1, GEN b1) assume $a_1/b_1 = 1 \bmod p$ if p odd and mod 2^4 if $p = 2$. Let $A_1 = a_1/p^v$ and $B_1 = b_1/p^v$ with $v = v_p(a_1) = v_p(b_1)$; let further $A_{n+1} = (A_n + B_n + 2B_{n+1})/4$, $B_{n+1} = B_n\sqrt{A_n/B_n}$ (the square root of A_nB_n congruent to B_n mod p) and $R_n = p^v(A_n - B_n)$. We stop when R_n is 0 at the given p-adic accuracy. This function returns in a triplet t_VEC the three sequences (A_n), (B_n) and (R_n), corresponding to a sequence of 2-isogenies on the Tate curve $y^2 = x(x - a_1)(x + a_1 - b_1)$. The common limit of A_n and B_n is the $M_2(a_1, b_1)$, the square of the p-adic AGM of $\sqrt{}(a_1)$ and $\sqrt{}(b_1)$. This is given by ellQp_Ei and is used by corresponding ascending and descending p-adic Landen transforms:

void Qp_ascending_Landen(GEN ABR, GEN *ptx, GEN *pty)

void Qp_descending_Landen(GEN ABR, GEN *ptx, GEN *pty)

10.10.5 Cached constants.

The cached constant is returned at its current precision, which may be larger than prec. One should always use the mp*xxx* variant: mppi, mpeuler, or mplog2.

GEN consteuler(long prec) precomputes Euler-Mascheroni's constant at precision prec.

GEN constcatalan(long prec) precomputes Catalan's constant at precision prec.

GEN constpi(long prec) precomputes π at precision prec.

GEN constlog2(long prec) precomputes $\log(2)$ at precision prec.

void constbern(long n) precomputes the n even Bernoulli numbers $B_2, \ldots, B_{2n}$ as t_FRAC. No more than n Bernoulli numbers will ever be stored (by bernfrac or bernreal), unless a subsequent call to constbern increases the cache.

GEN constzeta(long n, long prec) ensures that the n values $\gamma, \zeta(2), \ldots, \zeta(n)$ are cached at accuracy bigger than or equal to prec and return a vector containing at least those value. Note

that $\gamma = \lim_1 \zeta(s) - 1/(s-1)$. If the accuracy of cached data is too low or n is greater than the cache length, the cache is recomputed at the given parameters.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

`GEN mppi(long prec)` returns π at precision `prec`.

`GEN Pi2n(long n, long prec)` returns $2^n\pi$ at precision `prec`.

`GEN PiI2(long n, long prec)` returns the complex number $2\pi i$ at precision `prec`.

`GEN PiI2n(long n, long prec)` returns the complex number $2^n\pi i$ at precision `prec`.

`GEN mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

`GEN mpeuler(long prec)` returns Catalan's number at precision `prec`.

`GEN mplog2(long prec)` returns $\log 2$ at precision `prec`.

The following functions use the Bernoulli numbers cache initialized by `constbern`:

`GEN bernreal(long i, long prec)` returns the Bernoulli number B_i as a `t_REAL` at precision `prec`. If `constbern(n)` was called previously with $n \geq i$, then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed; the cache is not updated.

`GEN bernfrac(long i)` returns the Bernoulli number B_i as a rational number (`t_FRAC` or `t_INT`). If the `constbern` cache includes B_i, the latter is returned. Otherwise, the missing value is computed; the cache is not updated.

10.10.6 Obsolete functions.

`void mpbern(long n, long prec)`

10.11 Permutations .

Permutations are represented in two different ways

- (`perm`) a `t_VECSMALL` p representing the bijection $i \mapsto p[i]$; unless mentioned otherwise, this is the form used in the functions below for both input and output,
- (`cyc`) a `t_VEC` of `t_VECSMALL`s representing a product of disjoint cycles.

`GEN identity_perm(long n)` return the identity permutation on n symbols.

`GEN cyclic_perm(long n, long d)` return the cyclic permutation mapping i to $i + d \pmod n$ in S_n. Assume that $d \leq n$.

`GEN perm_mul(GEN s, GEN t)` multiply s and t (composition $s \circ t$)

`GEN perm_sqr(GEN s)` multiply s by itself (composition $s \circ s$)

`GEN perm_conj(GEN s, GEN t)` return sts^{-1}.

`int perm_commute(GEN p, GEN q)` return 1 if p and q commute, 0 otherwise.

`GEN perm_inv(GEN p)` returns the inverse of p.

`GEN perm_pow(GEN p, GEN n)` returns p^n

`GEN perm_powu(GEN p, ulong n)` returns p^n

`GEN cyc_pow_perm(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `perm`).

`GEN cyc_pow(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `cyc`).

`GEN perm_cycles(GEN p)` return the cyclic decomposition of p.

`GEN perm_order(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths).

`ulong perm_orderu(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths) assuming it fits in a `ulong`.

`long perm_sign(GEN p)` returns the sign of the permutation p.

`GEN vecperm_orbits(GEN gen, long n)` return the orbits of $\{1, 2, \ldots, n\}$ under the action of the subgroup of S_n generated by *gen*.

`GEN Z_to_perm(long n, GEN x)` as `numtoperm`, returning a `t_VECSMALL`.

`GEN perm_to_Z(GEN v)` as `permtonum` for a `t_VECSMALL` input.

`GEN perm_to_GAP(GEN p)` return a `t_STR` which is a representation of p comptatible with the GAP computer algebra system.

10.12 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group *grp* of order n is represented by its regular representation (for an arbitrary ordering of its element) in S_n. A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some S_n, where n is the multiple of the order. Such an n is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group *grp* is represented by a `t_VEC` with two components:

grp[1] is a generating subset $[s_1, \ldots, s_g]$ of *grp* expressed as a vector of permutations of length n.

grp[2] contains the relative orders $[o_1, \ldots, o_g]$ of the generators *grp*[1].

or alternatively, if it is not WSS, by the sorted list of its elements.

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal)` check whether *gal* is a small group or a Galois group. Returns the underlying small group.

`GEN checkgroupelts(GEN gal)` check whether *gal* is a small group or a Galois group, or a vector of permutations listing the group elements. Returns the list of group elements as permutations.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group *gal*.

`GEN cyclicgroup(GEN g, long s)` return the cyclic group with generator g of order s.

`GEN trivialgroup(void)` return the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators *g1*, *g2* with respecting relative orders *s1*, *s2*.

`GEN abelian_group(GEN v)` let v be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the `domain` of the *nontrivial* small group *grp*. Return an error if *grp* is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group *grp* of domain n as permutations.

`GEN groupelts_gen(GEN elts)`, where *elts* is the list of elements of a group, returns a small generating set.

`GEN groupelts_to_group(GEN elts)`, where *elts* is the list of elements of a group, returns the corresponding small group.

`GEN group_set(GEN grp, long n)` returns a *F2v b* such that $b[i]$ is set if and only if the small group *grp* of domain n contains a permutation sending 1 to i.

`GEN groupelts_set(GEN elts, long n)`, where *elts* is the list of elements of a small group of domain n, returns a *F2v b* such that $b[i]$ is set if and only if the small group contains a permutation sending 1 to i.

`GEN groupelts_conj_set(GEN elts, GEN p)`, where *elts* is the list of elements of a small group of domain n, returns a *F2v b* such that $b[i]$ is set if and only if the small group contains a permutation sending $p^{-1}[1]$ to $p^{-1}[i]$.

`int group_subgroup_is_faithful(GEN G, GEN H)` return 1 if the action of G on G/H by translation is faithful, 0 otherwise.

`GEN groupelts_conjclasses(GEN elts, long *pn)`, where *elts* is the list of elements of a small group (sorted with respect to `vecsmall_lexcmp`), return a `t_VECSMALL conj` of the same length such that `conj[i]` is the index in $\{1, \cdots, n\}$ of the conjugacy class of `elts[i]` for some unspecified but deterministic ordering of the classes, where n is the number of conjugacy classes. If `pn` is non `NULL`, `*pn` is set to n.

`GEN conjclasses_repr(GEN conj, long nb)`, where `conj` and `nb` are as returned by the call `groupelts_conjclasses(elts)`, return `t_VECSMALL` of length `nb` which gives the indices in `elts` of a representative of each conjugacy class.

`GEN group_to_cc(GEN G)`, where G is a small group or a Galois group, returns a `cc` (conjclasses) structure `[elts,conj,rep,flag]`, as obtained by `alggroupcenter`, where `conj` is `groupelts_conjclasses(elts)` and `rep` is the attached `conjclasses_repr`. `flag` is 1 if the permutation representation is transitive (in which case an element g of G is characterized by $g[1]$), and 0 otherwise. Shallow function.

`long group_order(GEN grp)` returns the order of the small group *grp* (which is the product of the relative orders).

`long group_istrivial(GEN grp)` returns 1 if the small group *grp* is of order 1, else 0.

`long group_isabelian(GEN grp)` returns 1 if the small group *grp* is Abelian, else 0.

`long group_iswss(GEN grp)` return 1 if the small group *grp* is WSS, else 0.

`GEN group_abelianHNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns the HNF matrix of *grp* with respect to the generating family *grp*[1]. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`GEN group_abelianSNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns its cyclic decomposition. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`long group_subgroup_isnormal(GEN G, GEN H)`, H being a subgroup of the small group G, returns 1 if H is normal in G, else 0.

`long group_isA4S4(GEN grp)` returns 1 if the small group *grp* is isomorphic to A_4, 2 if it is isomorphic to S_4, 3 if it is isomorphic to $(3 \times 3) : 4$ and 0 else. This is mainly to deal with the idiosyncrasy of the format.

`GEN group_leftcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the left coset gG as a vector of permutations.

`GEN group_rightcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the right coset Gg as a vector of permutations.

`long group_perm_normalize(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, return 1 if $gGg^{-1} = G$, else 0.

`GEN group_quotient(GEN G, GEN H)`, where G is a small group and H is a subgroup of G, returns the quotient map $G \to G/H$ as an abstract data structure.

`GEN groupelts_quotient(GEN elts, GEN H)`, where *elts* is the list of elements of a small group G, H is a subgroup of G, returns the quotient map $G \to G/H$ as an abstract data structure.

`GEN quotient_perm(GEN C, GEN g)` where C is the quotient map $G \to G/H$ for some subgroup H of G and g an element of G, return the image of g by C (i.e. the coset gH).

`GEN quotient_group(GEN C, GEN G)` where C is the quotient map $G \to G/H$ for some *normal* subgroup H of G, return the quotient group G/H as a small group.

`GEN quotient_groupelts(GEN C)` where C is the quotient map $G \to G/H$ for some group G and some *normal* subgroup H of G, return the list of elements of the quotient group G/H (as permutations over corresponding to the regular representation).

`GEN quotient_subgroup_lift(GEN C, GEN H, GEN S)` where C is the quotient map $G \to G/H$ for some group G normalizing H and S is a subgroup of G/H, return the inverse image of S by C.

`GEN group_subgroups(GEN grp)` returns the list of subgroups of the small group *grp* as a `t_VEC`.

`GEN groupelts_solvablesubgroups(GEN elts)` where *elts* is the list of elements of a finite group, returns the list of its solvable subgroups, each as a list of its elements.

`GEN subgroups_tableset(GEN S, long n)` where S is a vector of subgroups of domain n, returns a table which matchs the set of elements of the subgroups against the index of the subgroups.

`long tableset_find_index(GEN tbl, GEN set)` searchs the set `set` in the table `tbl` and returns its attached index, or 0 if not found.

`GEN groupelts_abelian_group(GEN elts)` where *elts* is the list of elements of an *Abelian* small group, returns the corresponding small group.

`long groupelts_exponent(GEN elts)` where *elts* is the list of elements of a small group, returns the exponent the group (the LCM of the order of the elements of the group).

`GEN groupelts_center(GEN elts)` where *elts* is the list of elements of a small group, returns the list of elements of the center of the group.

`GEN group_export(GEN grp, long format)` convert a small group to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

`GEN groupgen_export(GEN gen, long format)` convert a small group given by its generators to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

`long group_ident(GEN grp)` returns the index of the small group *grp* in the GAP4 Small Group library, see `galoisidentify`.

`long group_ident_trans(GEN grp)` returns the index of the regular representation of the small group *grp* in the GAP4 Transitive Group library, see `polgalois`.

Chapter 11:
Standard data structures

11.1 Character strings.

11.1.1 Functions returning a `char *`.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of s (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most n chars from s (uses `pari_malloc`). If s is longer than n, only n characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of s, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of s and t, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* uordinal(ulong x)` return the ordinal number attached to x (i.e. 1st, 2nd, etc.) as a `stack_malloc`'ed string.

`char* itostr(GEN x)` writes the `t_INT` x to a `stack_malloc`'ed string.

`char* GENtostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts x to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENtostr_raw(GEN x)` as `GENtostr` with the following differences: 1) the output format is `f_RAW`; 2) the result is allocated on the stack and *must not* be freed.

`char* GENtostr_unquoted(GEN x)` as `GENtostr_raw` with the following additional difference: a `t_STR` x is printed without enclosing quotes (to be used by `print`.

`char* GENtoTeXstr(GEN x)`, as `GENtostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` g being a vector of `GEN`s, returns a malloc'ed string, the concatenation of the `GENtostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

11.1.2 Functions returning a t_STR.

`GEN strtoGENstr(const char *s)` returns a `t_STR` with content s.

`GEN strntoGENstr(const char *s, long n)` returns a `t_STR` containing the first n characters of s.

`GEN chartoGENstr(char c)` returns a `t_STR` containing the character c.

`GEN GENtoGENstr(GEN x)` returns a `t_STR` containing the printed form of x (in `raw` format). This is often easier to use that `GENtostr` (which returns a malloc-ed `char*`) since there is no need to free the string after use.

`GEN GENtoGENstr_nospace(GEN x)` as `GENtoGENstr`, removing all spaces from the output.

`GEN Str(GEN g)` as `RgV_to_str` with output format `f_RAW`, but returns a `t_STR`, not a malloc'ed string.

`GEN strtex(GEN g)` as `RgV_to_str` with output format `f_TEX`, but returns a `t_STR`, not a malloc'ed string.

`GEN strexpand(GEN g)` as `RgV_to_str` with output format `f_RAW`, performing tilde and environment expansion on the result. Returns a `t_STR`, not a malloc'ed string.

`GEN gsprintf(const char *fmt, ...)` equivalent to `pari_sprintf(fmt,...`, followed by `strtoGENstr`. Returns a `t_STR`, not a malloc'ed string.

`GEN gvsprintf(const char *fmt, va_list ap)` variadic version of `gsprintf`

`GEN pari_base64(const char *s)` convert the string to base64 (RFC4648 "+/" with padding).

11.1.3 Dynamic strings.

A `pari_str` is a dynamic string which grows dynamically as needed. This structure contains private data and two public members `char *string`, which is the string itself and `use_stack` which tells whether the string lives

• on the PARI stack (value 1), meaning that it will be destroyed by any manipulation of the stack, e.g. a garbage collection or resetting `avma`;

• in malloc'ed memory (value 0), in which case it is impervious to stack manipulation but will need to be explicitly freed by the user after use, via `pari_free(s.string)`.

`void str_init(pari_str *S, int use_stack)` initializes a dynamic string; if `use_stack` is 0, then the string is malloc'ed, else it lives on the PARI stack.

`void str_printf(pari_str *S, const char *fmt, ...)` write to the end of S the remaining arguments according to PARI format `fmt`.

`void str_putc(pari_str *S, char c)` write the character c to the end of S.

`void str_puts(pari_str *S, const char *s)` write the string s to the end of S.

11.2 Output.

11.2.1 Output contexts.

An output coutext, of type `PariOUT`, is a `struct` that models a stream and contains the following function pointers:

```
void (*putch)(char);           /* fputc()-alike */
void (*puts)(const char*);     /* fputs()-alike */
void (*flush)(void);           /* fflush()-alike */
```

The methods `putch` and `puts` are used to print a character or a string respectively. The method `flush` is called to finalize a messages.

The generic functions `pari_putc`, `pari_puts`, `pari_flush` and `pari_printf` print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

`void out_putc(PariOUT *out,  char c)` essentially equivalent to `out->putc(c)`. In addition, registers whether the last character printed was a `\n`.

`void out_puts(PariOUT *out, const char *s)` essentially equivalent to `out->puts(s)`. In addition, registers whether the last character printed was a `\n`.

`void out_printf(PariOUT *out, const char *fmt, ...)`

`void out_vprintf(PariOUT *out, const char *fmt, va_list ap)`

N.B. The function `out_flush` does not exist since it would be identical to `out->flush()`

`int pari_last_was_newline(void)` returns a nonzero value if the last character printed via `out_putc` or `out_puts` was `\n`, and 0 otherwise.

`void pari_set_last_newline(int last)` sets the boolean value to be returned by the function `pari_last_was_newline` to *last*.

11.2.2 Default output context. They are defined by the global variables `pariOut` and `pariErr` for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in nontrivial ways, this probably means that you are rewriting `gp`. For completeness, we document in this section what the default output contexts do.

pariOut. writes output to the `FILE*` `pari_outfile`, initialized to `stdout`. The low-level methods are actually the standard `putc` / `fputs`, plus some magic to handle a log file if one is open.

pariErr. prints to the `FILE*` `pari_errfile`, initialized to `stderr`. The low-level methods are as above.

You can stick with the default `pariOut` output context and change PARI's standard output, redirecting `pari_outfile` to another file, using

`void switchout(const char *name)` where `name` is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to `stdout`, call `switchout(NULL)`.

11.2.3 PARI colors. In this section we describe the low-level functions used to implement GP's color scheme, attached to the `colors` default. The following symbolic names are attached to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a nonempty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

11.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

```
void brute(GEN x, char format, long dec)

void matbrute(GEN x, char format, long dec)

void texe(GEN x, char format, long dec)
```

11.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if the debug level for `"files"` is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file f is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named s. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read n chars from stream f, storing the result in pre-allocated buffer b (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on s. Returns a `malloc`'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on s, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(but, max, s, localtime(&t);
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or `NULL` if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

11.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if the debug level for `"files"` is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return `NULL` on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is s, and associates a (read-only) `pariFILE` with it. If s is a compressed file (`.gz` suffix), it is uncompressed on the fly. If s cannot be opened, also try to open s`.gz`. Returns `NULL` on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safefopen(const char *s, const char *mode)` creates a *new* file s (a priori for writing) with `600` permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

11.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from an environment variable (`$GPTMPDIR`, else `$TMPDIR`), or the first writable directory among `/tmp`, `/var/tmp` and `.`.

`char* pari_unique_dir(const char *s)` creates a "unique directory" and return its name built from the string s, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is `malloc`'ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string s (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name and creates an empty file with that name. The name returned is `malloc`'ed.

`char* pari_unique_filename_suffix(const char *s, const char *suf)` analogous to above `pari_unique_filename`, creating a (previously nonexistent) tempfile whose name ends with suffix `suf`.

11.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

```
void pari_err(numerr, ...)
```

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where `fmt` is a PARI format, producing a string s from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_`*ERRORCLASS*`(...)` instead of `pari_err(e_`*ERRORCLASS*`, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

11.4.1 Internal errors, "system" errors.

11.4.1.1 e_ARCH. A requested feature s is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, 's' not available on this system`.

11.4.1.2 e_BUG. A bug in the PARI library, in function s.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in `s`, please report`.

11.4.1.3 e_FILE. Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: `error opening` *what*: ‘*name*’.

11.4.1.4 e_FILEDESC. Error while handling a file descriptor.

```
pari_err(e_FILEDESC, const char *where, long n)
pari_err_FILEDESC(const char *where, long n)
```

prints the error message: `invalid file descriptor in` *where*: ‘*name*’.

11.4.1.5 e_IMPL. A requested feature s is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: `sorry,` s `is not yet implemented`.

11.4.1.6 e_PACKAGE. Missing optional package s.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: `package` s `is required, please install it`

11.4.2 Syntax errors, type errors.

11.4.2.1 e_DIM. arguments submitted to function s have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a nonsquare matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: `inconsistent dimensions in` s.

11.4.2.2 e_FLAG. A flag argument is out of bounds in function s.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: `invalid flag in` s.

11.4.2.3 e_NOTFUNC. Generated by the PARI evaluator; tried to use a `GEN` which is not a `t_CLOSURE` in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: `not a function in a function call`.

11.4.2.4 e_OP. Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in `O(2) + O(3)`: it is valid to add two `t_PADIC`s, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As `e_TYPE2`, replacing `forbidden` by `inconsistent`.

11.4.2.5 e_PRIORITY. object o in function s contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y)`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in` s`, variable` v_o op v, were v_o is `gvar(o)`.

11.4.2.6 e_SYNTAX. Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry` $\leq$ `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e` $-$ `entry` is large); a caret points at `e`, indicating where the error took place.

11.4.2.7 e_TYPE. An argument x of function s had an unexpected type. (As in `factor("blah")`.)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in` s `(t_`x`)`, where `t_`x is the type of x.

11.4.2.8 e_TYPE2. Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi`.)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden` s `t_`x op `t_`y, where `t_`z denotes the type of z. Here, s denotes the spelled out name of the operator $op \in \{+, *, /, \%, =\}$, e.g. *addition* for `"+"` or *assignment* for `"="`. If op is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. `"gcd"`, and the error message becomes `forbidden` op `t_`x, `t_`y.

11.4.2.9 e_VAR. polynomials x and y submitted to function s have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in` s X `!=` Y, where X and Y are the names of the variables of x and y, respectively.

11.4.3 Overflows.

11.4.3.1 e_COMPONENT. Trying to access an inexistent component of a vector/matrix/list: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `nonexistent component in` f`:  index` op lim. Special case: if f is the empty string (no meaningful public function name can be used), we ignore it and print the message: `nonexistent component:  index` op lim.

11.4.3.2 e_DOMAIN. An argument x is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in` f`:` v op lim. Special case: if op is the empty string, we ignore lim and print the error message: `domain error in` f`:` v `out of range`.

11.4.3.3 e_MAXPRIME. A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~`c if c is nonzero. And simply `not enough precomputed primes` otherwise.

11.4.3.4 e_MEM. A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

11.4.3.5 e_OVERFLOW. An object in function s becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in` s.

11.4.3.6 e_PREC. Function s fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in` s.

11.4.3.7 e_STACK. The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

11.4.4 Errors triggered intentionally.

11.4.4.1 e_ALARM. A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: s.

11.4.4.2 e_USER. A user error, as triggered by `error`$(g_1, \ldots, g_n)$ in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector g.

11.4.5 Mathematical errors.

11.4.5.1 e_CONSTPOL. An argument of function s is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in` s.

11.4.5.2 e_COPRIME. Function s expected two coprime arguments, and did receive x, y which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in` s`:` x, y.

11.4.5.3 e_INV. Tried to invert a noninvertible object x.

```
pari_err(e_INV, const char *s, GEN x)
pari_err_INV(const char *s, GEN x)
```

prints the error message: `impossible inverse in` s`:` x. If $x = \texttt{Mod}(a, b)$ is a `t_INTMOD` and a is not 0 mod b, this allows to factor the modulus, as $\texttt{gcd}(a, b)$ is a nontrivial divisor of b.

11.4.5.4 e_IRREDPOL. Function s expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in` s`:` x.

11.4.5.5 e_MISC. Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: s.

11.4.5.6 e_MODULUS. moduli x and y submitted to function s are inconsistent. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in` s, then the moduli.

11.4.5.7 e_PRIME. Function s expected a prime number, and did receive p, which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in` s`:` x.

11.4.5.8 e_ROOTS0. An argument of function s is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in` s.

11.4.5.9 e_SQRTN. Tried to compute an n-th root of x, which does not exist, in function s. (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQRTN, GEN x)
pari_err_SQRTN(GEN x)
```

prints the error message: `not an n-th power residue in` s`:` x.

11.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR err`. The name is allocated on the PARI stack and must not be freed.

`int pari_err_display(GEN err)` displays the error message corresponding to `err`. Always return 0. Default value of the callback `cb_pari_err_handle`.

11.5 Hashtables.

A `hashtable`, or associative array, is a set of pairs (k, v) of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or `hashentry`, contains

• a key/value pair (k, v), both of type `void*` for maximal flexibility,

• the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,

• a link pointer to the next entry sharing the same table cell.

```
typedef struct {
  void *key, *val;
  ulong hash; /* hash(key) */
  struct hashentry *next;
```

```
} hashentry;

typedef struct {
  ulong len; /* table length */
  hashentry **table; /* the table */
  ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
  ulong pindex; /* prime index */
  ulong (*hash) (void *k); /* hash function */
  int (*eq) (void *k1, void *k2); /* equality test */
  int use_stack; /* use the PARI stack, resp. malloc */
} hashtable;
```

```
hashtable* hash_create(size, hash, eq, use_stack)
    ulong size;
    ulong (*hash)(void*);
    int (*eq)(void*,void*);
    int use_stack;
```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`hashtable* hash_create_GEN(ulong size, long stack)` special case when the keys are `GEN` with `===` (`gidentical`) equality test.

`hashtable* hash_create_INT(ulong size, long stack)` special case when the keys are `t_INT` with ordinary equality test.

`hashtable* hash_create_ulong(ulong size, long stack)` special case when the keys are `ulongs` with ordinary equality test.

`hashtable* hash_create_str(ulong size, long stack)` special case when the keys are character strings with string equality test (and `hash_str` hash function).

`void hash_init(hashtable *h, ulong size, ulong (*hash)(void*), int (*eq)(void*, void*), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `void*`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_init_GEN(hashtable *h, ulong size, int (*eq)(GEN, GEN), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `GEN`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`. The hash used is `hash_GEN`.

`void hash_init_ulong(hashtable *h, ulong size, use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `ulong`. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts (k, v) in hashtable h. No copy is made: k and v themselves are stored. The implementation does not prevent one to insert two entries with equal keys k, but which of the two is affected by later commands is undefined.

`void hash_insert2(hashtable *h, void *k, void *v, ulong hash)` as `hash_insert`, assuming `h->hash(k)` is `hash`.

`void hash_insert_long(hashtable *h, void *k, long v)` as `hash_insert` but `v` is a `long`.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key k in h. Return it if it one exists, and `NULL` if not.

`hashentry* hash_search2(hashtable *h, void *k, ulong hash)` as `hash_search` assuming `h->hash(k)` is `hash`.

`GEN hash_haskey_GEN(hashtable *h, void *k)` returns the associate value if the key k belongs to the hash, otherwise returns `NULL`.

`int hash_haskey_long(hashtable *h, void *k, long *v)` returns 1 if the key k belongs to the hash and set v to its value, otherwise returns 0.

`hashentry * hash_select(hashtable *h, void *k, void *E, int (*select)(void *, hashentry *))` variant of `hash_search`, useful when entries with identical keys are inserted: among the entries attached to key k, return one satisfying the selection criterion (such that `select(E,e)` is nonzero), or `NULL` if none exist.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry (k, v) with key k from h and return it. (Return `NULL` if none was found.) Only the linking structures are freed, memory attached to k and v is not reclaimed.

`hashentry* hash_remove_select(hashtable *h, void *k, void *E, int(*select)(void*, hashentry *))` a variant of `hash_remove`, useful when entries with identical keys are inserted: among the entries attached to key k, return one satisfying the selection criterion (such that `select(E,e)` is nonzero) and delete it, or `NULL` if none exist. Only the linking structures are freed, memory attached to k and v is not reclaimed.

`GEN hash_keys(hashtable *h)` return in a `t_VECSMALL` the keys stored in hashtable h.

`GEN hash_keys_GEN(hashtable *h)` return in a `t_VEC` the keys stored in hashtable h (which are assumed to be `GEN`s).

`GEN hash_values(hashtable *h)` return in a `t_VECSMALL` the values stored in hashtable h.

`GEN hash_values_GEN(hashtable *h)` return in a `t_VEC` the values stored in hashtable h (which are assumed to be `GEN`s).

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

`void hash_dbg(hashtable *h)` print statistics for hashtable h, allows to evaluate the attached hash function performance on actual data.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str_len(const char *s, long len)` hash the prefix string containing the first `len` characters (assume `strlen`$(s) \geq$ `len`).

`ulong hash_GEN(GEN x)` generic hash function. Not suitable for cryptographic use. The value returned is different on 32 and 64 bit architectures but it does not depend on the multiprecision kernel (whether GMP is used or not).

`ulong hash_zv(GEN x)` hash a `t_VECSMALL`.

11.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

11.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)&t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

11.6.2 Adding elements. The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
  long n = pari_stack_new(&s_foo);
  t_foo[n] = x;
}
```

11.6.3 Accessing elements.

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
  foo x;
  if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
  x        = t_foo[a];
  t_foo[a] = t_foo[b];
  t_foo[b] = x;
}
```

11.6.4 Stack of stacks. Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

11.6.5 Public interface. Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

- `s.alloc` is the number of elements allocated for `data`.
- `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as $0 \leq$ `s.n` $\leq$ `s.alloc` holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that `s.alloc` $\geq$ `s.n` $+$ `nb`, possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n` -1.

Caveat. The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

11.7 Vectors and Matrices.

11.7.1 Access and extract. See Section 9.2.1 and Section 9.2.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, all of them are shallow functions. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a `GEN` whose components are the components of x (no copy is made). The result may now be used to compute in place without destroying x. This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for nonrecursive types, but is useless in that case since it makes a deep copy. If x is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if x is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where x is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of x, *without* copying its components, i. e., it returns a `GEN` whose components are (physically) the components of x. This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate x and y, *without* copying components, i. e., it returns a `GEN` whose components are (physically) the components of x and y.

`GEN shallowconcat1(GEN x)` x must be `t_VEC`, `t_COL` or `t_LIST`, concatenate its elements from left to right. Shallow version of `gconcat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN RgM_flatten_RgC(GEN x)` concatenates the columns of x in order. Shallow.

`GEN RgM_flatten_RgV(GEN x)` concatenates the rows of x in order. Shallow.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix x according to the selection parameter y. This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN RgV_F2v_extract_shallow(GEN V, GEN x)` extract components of the vector V whose indices corresponds to non-zero components of x. Shallow function.

`GEN shallowmatextract(GEN M, GEN l1, GEN l2)` extract components of the matrix M according to the `t_VECSMALL` $l1$ (list of lines indices) and $l2$ (list of columns indices). This is the shallow analog of `extract0(x, l1, l2)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT` A, return the matrix with i-th row and j-th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT` A and B of compatible dimensions. A `NULL` pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN matslice(GEN A, long a, long b, long c, long d)` returns the submatrix $A[a..b, c..d]$. Assume $a \leq b$ and $c \leq d$.

`GEN row(GEN A, long i)` return $A[i,]$, the i-th row of the `t_MAT` A.

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the i-th row of `t_MAT` A: $A[i, j_1]$, $A[i, j_1 + 1] \ldots, A[i, j_2]$. Assume $j_1 \leq j_2$.

`GEN rowcopy(GEN A, long i)` return the row $A[i,]$ of the `t_MAT` A. This function is stack-clean. See `row` for the shallow equivalent.

`GEN rowslice(GEN A, long i1, long i2)` return the `t_MAT` formed by the i_1-th through i_2-th rows of `t_MAT` A. Assume $i_1 \leq i_2$.

`GEN rowsplice(GEN A, long i)` return the `t_MAT` formed from the coefficients of `t_MAT` A with j-th row removed.

`GEN rowpermute(GEN A, GEN p)`, p being a `t_VECSMALL` representing a list $[p_1, \ldots, p_n]$ of rows of `t_MAT` A, returns the matrix whose rows are $A[p_1,], \ldots, A[p_n,]$.

`GEN rowslicepermute(GEN A, GEN p, long x1, long x2)`, short for

```
rowslice(rowpermute(A,p), x1, x2)
```

(more efficient).

`GEN vecslice(GEN A, long j1, long j2)`, return $A[j_1], \ldots, A[j_2]$. If A is a `t_MAT`, these correspond to *columns* of A. The object returned has the same type as A (`t_VECSMALL`, `t_VEC`, `t_COL` or `t_MAT`). Assume $j_1 \leq j_2$ or $j_2 = j_1 - 1$ (return empty vector/matrix).

`GEN vecsplice(GEN A, long j)` return A with j-th entry removed (`t_VEC`, `t_COL`) or j-th column removed (`t_MAT`).

`GEN veclast(GEN A)` return the last entry of A (`t_VEC`, `t_COL`) or last column of A (`t_MAT`). Shallow, undefined if `lg(A)` is 1.

`GEN vecreverse(GEN A)`. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are the $A[n], \dots, A[1]$. If A is a `t_MAT`, these are the *columns* of A.

`void vecreverse_inplace(GEN A)` as `vecreverse`, but reverse A in place.

`GEN vecpermute(GEN A, GEN p)` p is a `t_VECSMALL` representing a list $[p_1, \dots, p_n]$ of indices. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are $A[p_1], \dots, A[p_n]$. If A is a `t_MAT`, these are the *columns* of A.

`GEN vecsmallpermute(GEN A, GEN p)` as `vecpermute` when `A` is a `t_VECSMALL`.

`GEN vecslicepermute(GEN A, GEN p, long y1, long y2)` short for

```
vecslice(vecpermute(A,p), y1, y2)
```

(more efficient).

11.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

```
for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))
```

for suitable f:

`GEN vecinv(GEN a)`. Given a vector a, returns the vector whose i-th component is `ginv`$(a[i])$.

`GEN vecmul(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gmul`$(a[i], b[i])$.

`GEN vecdiv(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gdiv`$(a[i], b[i])$.

`GEN vecsqr(GEN a)` returns the vector whose i-th component is `gsqr`$(a[i])$.

`GEN vecpow(GEN a, GEN n)`. Given n a `t_INT`, returns the vector whose i-th component is $a[i]^n$.

`GEN vecmodii(GEN a, GEN b)`. Assuming a and b are two `ZV` of the same length, returns the vector whose i-th component is `modii`$(a[i], b[i])$.

`GEN vecmoduu(GEN a, GEN b)`. Assuming a and b are two `t_VECSMALL` of the same length, returns the vector whose i-th component is $a[i] \ \% \ b[i]$.

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

11.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of `GEN`s. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

Note. All the functions below are shallow.

`GEN const_col(long n, GEN x)` returns a `t_COL` of `n` components equal to `x`.

`GEN const_vec(long n, GEN x)` returns a `t_VEC` of `n` components equal to `x`.

`int vec_isconst(GEN v)` Returns 1 if all the components of `v` are equal, else returns 0.

`void vec_setconst(GEN v, GEN x)` v a pre-existing vector. Set all its components to x.

`int vec_is1to1(GEN v)` Returns 1 if the components of `v` are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

`GEN vec_append(GEN V, GEN s)` append `s` to the vector `V`.

`GEN vec_prepend(GEN V, GEN s)` prepend `s` to the vector `V`.

`GEN vec_shorten(GEN v, long n)` shortens the vector `v` to `n` components.

`GEN vec_lengthen(GEN v, long n)` lengthens the vector `v` to `n` components. The extra components are not initialized.

`GEN vec_insert(GEN v, long n, GEN x)` inserts x at position n in the vector v.

`GEN vec_equiv(GEN O)` given a vector of objects O, return a vector with n components where n is the number of distinct objects in O. The i-th component is a `t_VECSMALL` containing the indices of the elements in O having the same value. Applied to the image of a function evaluated on some finite set, it computes the fibers of the function.

`GEN vec_reduce(GEN O, GEN *pE)` given a vector of objects O, return the vector v (of the same type as O) of *distinct* elements of O and set a `t_VECSMALL` E with the same length as v, such that $E[i]$ is the multiplicity of object $v[i]$ in the original O. Shallow function.

11.8 Vectors of small integers.

11.8.1 `t_VECSMALL`.

These functions handle `t_VECSMALL` as an abstract container type of small signed integers. No specific meaning is attached to the content.

`GEN const_vecsmall(long n, long c)` returns a `t_VECSMALL` of `n` components equal to `c`.

`GEN vec_to_vecsmall(GEN z)` identical to `ZV_to_zv(z)`.

`GEN vecsmall_to_vec(GEN z)` identical to `zv_to_ZV(z)`.

`GEN vecsmall_to_col(GEN z)` identical to `zv_to_ZC(z)`.

`GEN vecsmall_to_vec_inplace(GEN z)` apply `stoi` to all entries of z and set its type to `t_VEC`.

`GEN vecsmall_copy(GEN x)` makes a copy of `x` on the stack.

`GEN vecsmall_shorten(GEN v, long n)` shortens the `t_VECSMALL` `v` to `n` components.

`GEN vecsmall_lengthen(GEN v, long n)` lengthens the `t_VECSMALL` `v` to `n` components. The extra components are not initialized.

`GEN vecsmall_indexsort(GEN x)` performs an indirect sort of the components of the `t_VECSMALL` `x` and return a permutation stored in a `t_VECSMALL` (merge sort).

`void vecsmall_sort(GEN v)` sorts the `t_VECSMALL` `v` in place (merge sort).

`GEN vecsmall_reverse(GEN v)` as `vecreverse` for a `t_VECSMALL` `v`.

`long vecsmall_max(GEN v)` returns the maximum of the elements of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_indexmax(GEN v)` returns the index of the largest element of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_min(GEN v)` returns the minimum of the elements of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_indexmin(GEN v)` returns the index of the smallest element of `t_VECSMALL` `v`, assumed nonempty.

`int vecsmall_isconst(GEN v)` Returns 1 if all the components of `v` are equal, else returns 0.

`int vecsmall_is1to1(GEN v)` Returns 1 if the components of `v` are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

`long vecsmall_isin(GEN v, long x)` returns the first index i such that `v[`i`]` is equal to `x`. Naive search in linear time, does not assume that `v` is sorted.

`GEN vecsmall_uniq(GEN v)` given a `t_VECSMALL` `v`, return the vector of unique occurrences.

`GEN vecsmall_uniq_sorted(GEN v)` same as `vecsmall_uniq`, but assumes `v` sorted.

`long vecsmall_duplicate(GEN v)` given a `t_VECSMALL` `v`, return 0 if there is no duplicates, or the index of the first duplicate (`vecsmall_duplicate([1,1])` returns 2).

`long vecsmall_duplicate_sorted(GEN v)` same as `vecsmall_duplicate`, but assume `v` sorted.

`int vecsmall_lexcmp(GEN x, GEN y)` compares two `t_VECSMALL` lexically.

`int vecsmall_prefixcmp(GEN x, GEN y)` truncate the longest `t_VECSMALL` to the length of the shortest and compares them lexicographically.

`GEN vecsmall_prepend(GEN V, long s)` prepend `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_append(GEN V, long s)` append `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_concat(GEN u, GEN v)` concat the `t_VECSMALL` `u` and `v`.

`long vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where `u` and `v` agree.

`long vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL` `v` as the digit of a number in base `base` and return this number modulo `mod`. This can be used as an hash function.

`GEN vecsmall_prod(GEN v)` given a `t_VECSMALL` `v`, return the product of its entries.

The following sorting functions assume all entries of the `t_VECSMALL` v satisfy $0 \leq v[i] \leq M$ and use a counting sort, in linear time $O(\#v + M)$:

`void vecsmall_counting_sort(GEN v, long M)` sorts the `t_VECSMALL` `v` in place.

`GEN vecsmall_counting_indexsort(GEN v, long M)` as `vecsmall_indexsort` using a counting sort.

`GEN vecsmall_counting_uniq(GEN v, long M)` as `vecsmall_uniq` using a counting sort.

11.8.2 Vectors of `t_VECSMALL`. These functions manipulate vectors of `t_VECSMALL` (vecvecsmall).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_shallow(GEN x)`, shallow variant of `vecvecsmall_sort`.

`void vecvecsmall_sort_inplace(GEN x, GEN *perm)` sort lexicographically `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y)` `x` being a sorted vecvecsmall and `y` a `t_VECSMALL`, search `y` inside `x`.

`GEN vecvecsmall_max(GEN x)` returns the largest entry in all $x[i]$, assumed nonempty. Shallow function.

Chapter 12:
Functions related to the GP interpreter

12.1 Handling closures.

12.1.1 Functions to evaluate t_CLOSURE.

`void closure_disassemble(GEN C)` print the `t_CLOSURE` `C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE` `C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity $\geq$ `n`.

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`. Assumes `C` has arity $\geq$ `lg(args)` $-$ `1`.

`GEN closure_callgenvecprec(GEN C, GEN args, long prec)` as `closure_callgenvec` but set the precision locally to `prec`.

`GEN closure_callgenvecdef(GEN C, GEN args, GEN def)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`, where the `t_VECSMALL` `def` indicates which arguments are actually present. Assumes `C` has arity $\geq$ `lg(args)` $-$ `1`.

`GEN closure_callgenvecdefprec(GEN C, GEN args, GEN def, long prec)` as `closure_callgenvecdef` but set the precision locally to `prec`.

`GEN closure_callgen0(GEN C, long prec)` evaluate the `t_CLOSURE` `C` without arguments.

`GEN closure_callgen0prec(GEN C, long prec)` evaluate the `t_CLOSURE` `C` without arguments, but set the precision locally to `prec`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x`. Assumes `C` has arity ≥ 1.

`GEN closure_callgen1prec(GEN C, GEN x, long prec)` as `closure_callgen1`, but set the precision locally to `prec`.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE` `C` with argument `x`, `y`. Assumes `C` has arity ≥ 2.

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x` and discard the result. Assumes `C` has arity ≥ 1.

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (break, next and return) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

- raises an exception (`closure_evalnobrk`).
- passes through (by returning NULL itself).

- handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates an inline closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates an inline closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

`GEN closure_evalres(GEN code)` evaluates an inline closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

`GEN closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a nonzero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

12.1.2 Control flow functions.

```
GEN derivnum0(GEN a, GEN code, GEN ind, long prec)

GEN direuler0(GEN a, GEN b, GEN code, GEN c)

GEN dirpowerssum0(GEN N, GEN s, GEN code, long both, long prec)

void fordiv(GEN a, GEN code)

void fordivfactored(GEN a, GEN code)

void foreachpari(GEN a, GEN code)

void forell0(long a, long b, GEN code, long flag)

void forfactored(GEN a, GEN b, GEN code)

void forpari(GEN a, GEN b, GEN code)

void forpart0(GEN k, GEN code, GEN nbound, GEN abound)

void forperm0(GEN k, GEN code)

void forsquarefree(GEN a, GEN b, GEN code)

void forstep(GEN a, GEN b, GEN s, GEN code)

void forsubgroup0(GEN cyc, GEN bound, GEN code)

void forsubset0(GEN nk, GEN code)

GEN iferrpari(GEN a, GEN b, GEN c)

GEN ifpari(GEN g, GEN a, GEN b)

GEN intcirc0(GEN a, GEN R, GEN code, GEN tab, long prec)

GEN intfuncinit0(GEN a, GEN b, GEN code, long m, long prec)

GEN intnum0(GEN a, GEN b, GEN code, GEN tab, long prec)
```

`GEN laurentseries0(GEN code, long M, long v, long prec)`

`void pari_breakpoint(void)`

`void pariplot0(GEN a, GEN b, GEN code, GEN ysmlu, GEN ybigu, long prec)`

`GEN ploth0(GEN a, GEN b, GEN code, long flag, long n, long prec)`

`GEN plothexport0(GEN fmt, GEN a, GEN b, GEN code, long flags, long n, long prec)`

`GEN prodeuler0(GEN a, GEN b, GEN code, long prec)`

`GEN prodinf0(GEN a, GEN code, long flag, long prec)`

`GEN solvestep0(GEN a, GEN b, GEN step, GEN code, long flag, long prec)`

`GEN sumalt0(GEN a, GEN code, long flag, long prec)`

`GEN sumdivmultexpr0(GEN num, GEN code)`

`GEN suminf0(GEN a, GEN code, long bit)`

`GEN sumnum0(GEN a, GEN code, GEN tab, long prec)`

`GEN sumnumap0(GEN a, GEN code, GEN tab, long prec)`

`GEN sumnumlagrange0(GEN a, GEN code, GEN tab, long prec)`

`GEN sumpos0(GEN a, GEN code, long flag, long prec)`

`void untilpari(GEN a, GEN code)`

`void whilepari(GEN a, GEN code)`

`GEN zbrent0(GEN a, GEN b, GEN code, long prec)`

12.1.3 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

12.1.4 Functions to deal with lexical local variables.

Function using the prototype code `'V'` need to manually create and delete a lexical variable for each code `'V'`, which will be given a number $-1, -2, \ldots$.

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is a on the top of the stack. This variable get the number -1, and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the n topmost lexical variables, increasing the number of other variables by n. The argument n must match the number of variables allocated through `push_lex`.

`GEN get_lex(long vn)` get the value of the variable with number `vn`.

`void set_lex(long vn, GEN x)` set the value of the variable with number `vn`.

12.1.5 Functions returning new closures.

`GEN compile_str(const char *s)` returns the closure corresponding to the GP expression s.

`GEN closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure `code`.

`GEN closure_derivn(GEN code, long n)` returns a closure corresponding to the numerical derivative of order $n > 0$ of the closure `code`.

`GEN closuretoinl(GEN code)` convert the closure `code` to a pseudo-inline closure suitable for use with prototype codes `I` and `E`.

`GEN snm_closure(entree *ep, GEN data)` Let `data` be a vector of length m, `ep` be an `entree` pointing to a C function f of arity $n + m$, returns a `t_CLOSURE` object g of arity n such that $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, gel(data, 1), ..., gel(data, m))$. If `data` is `NULL`, then $m = 0$ is assumed. Shallow function.

`GEN strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

`GEN strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the n last parameters set to the n `GEN`s following n. This is analogous to `snm_closure(is_entry(str), mkvecn(...))` but the latter has lower overhead since it does not copy arguments, nor does it validate inputs.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

12.1.6 Functions used by the gp debugger (break loop). `long closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

`void closure_err(long level)` prints a backtrace of the last 20 stack frames, starting at frame `level`, the numbering starting at 0.

12.1.7 Standard wrappers for iterators. Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

12.1.7.1 Standard wrappers for inline closures. These wrappers are used to implement GP functions taking inline closures as input. The object `(GEN)E` must be an inline closure which is evaluated with the lexical variable number -1 set to x.

`GEN gp_eval(void *E, GEN x)` is used for the prototype code `'E'`.

`GEN gp_evalprec(void *E, GEN x, long prec)` as `gp_eval`, but set the precision locally to `prec`.

`long gp_evalvoid(void *E, GEN x)` is used for the prototype code `'I'`. The resulting value is discarded. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

`long gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is nonzero).

`GEN gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

12.1.7.2 Standard wrappers for true closures. These wrappers are used to implement GP functions taking true closures as input.

`GEN gp_call(void *E, GEN x)` evaluates the closure `(GEN)E` on x.

`GEN gp_callprec(void *E, GEN x, long prec)` as `gp_call`, but set the precision locally to `prec`.

`GEN gp_call2(void *E, GEN x, GEN y)` evaluates the closure `(GEN)E` on (x, y).

`long gp_callbool(void *E, GEN x)` evaluates the closure `(GEN)E` on x, returns `1` if its result is nonzero, and `0` otherwise.

`long gp_callvoid(void *E, GEN x)` evaluates the closure `(GEN)E` on x, discarding the result. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

12.2 Defaults.

`entree* pari_is_default(const char *s)` return the `entree` structure attached to s if it is the name of a default, `NULL` otherwise.

`GEN setdefault(const char *s, const char *v, long flag)` is the low-level function underlying `default0`. If s is `NULL`, call all default setting functions with string argument `NULL` and flag `d_ACKNOWLEDGE`. Otherwise, check whether s corresponds to a default and call the corresponding default setting function with arguments v and *flag*.

We shall describe these functions below: if v is `NULL`, we only look at the default value (and possibly print or return it, depending on `flag`); otherwise the value of the default to v, possibly after some translation work. The flag is one of

- `d_INITRC` called while reading the `gprc`: print and return `gnil`, possibly defer until `gp` actually starts.
- `d_RETURN` return the current value, as a `t_INT` if possible, as a `t_STR` otherwise.
- `d_ACKNOWLEDGE` print the current value, return `gnil`.
- `d_SILENT` print nothing, return `gnil`.

Low-level functions called by `setdefault`:

`GEN sd_TeXstyle(const char *v, long flag)`

`GEN sd_breakloop(const char *v, long flag)`

`GEN sd_colors(const char *v, long flag)`

`GEN sd_compatible(const char *v, long flag)`

`GEN sd_datadir(const char *v, long flag)`

`GEN sd_debug(const char *v, long flag)`

`GEN sd_debugfiles(const char *v, long flag)`

`GEN sd_debugmem(const char *v, long flag)`

`GEN sd_doctest(const char *v, long flag)`

`GEN sd_echo(const char *v, long flag)`

GEN sd_factor_add_primes(const char *v, long flag)

GEN sd_factor_proven(const char *v, long flag)

GEN sd_factorlimit(const char *v, long flag)

GEN sd_format(const char *v, long flag)

GEN sd_graphcolormap(const char *v, long flag)

GEN sd_graphcolors(const char *v, long flag)

GEN sd_help(const char *v, long flag)

GEN sd_histfile(const char *v, long flag)

GEN sd_histsize(const char *v, long flag)

GEN sd_lines(const char *v, long flag)

GEN sd_linewrap(const char *v, long flag)

GEN sd_log(const char *v, long flag)

GEN sd_logfile(const char *v, long flag)

GEN sd_nbthreads(const char *v, long flag)

GEN sd_new_galois_format(const char *v, long flag)

GEN sd_output(const char *v, long flag)

GEN sd_parisize(const char *v, long flag)

GEN sd_parisizemax(const char *v, long flag)

GEN sd_path(const char *v, long flag)

GEN sd_plothsizes(const char *v, long flag)

GEN sd_prettyprinter(const char *v, long flag)

GEN sd_primelimit(const char *v, long flag)

GEN sd_prompt(const char *v, long flag)

GEN sd_prompt_cont(const char *v, long flag)

GEN sd_psfile(const char *v, long flag) The `psfile` default is obsolete, don't use this function.

GEN sd_readline(const char *v, long flag)

GEN sd_realbitprecision(const char *v, long flag)

GEN sd_realprecision(const char *v, long flag)

GEN sd_recover(const char *v, long flag)

GEN sd_secure(const char *v, long flag)

GEN sd_seriesprecision(const char *v, long flag)

GEN sd_simplify(const char *v, long flag)

GEN sd_sopath(const char *v, int flag)

GEN sd_strictargs(const char *v, long flag)

GEN sd_strictmatch(const char *v, long flag)

GEN sd_timer(const char *v, long flag)

GEN sd_threadsize(const char *v, long flag)

GEN sd_threadsizemax(const char *v, long flag)

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

• v and flag are the arguments passed to default: v is a new value (or the empty string: no change), and flag is one of d_INITRC, d_RETURN, etc.

• s is the name of the default being changed, used to display error messages or acknowledgements.

GEN sd_toggle(const char *v, long flag, const char *s, int *ptn)

• if v is neither "0" nor "1", an error is raised using pari_err.

• ptn points to the current numerical value of the toggle (1 or 0), and is set to the new value (when v is nonempty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on flag:

• d_RETURN: returns the new toggle value, as a GEN.

• d_ACKNOWLEDGE: prints a message indicating the new toggle value and return gnil.

• other cases: print nothing and return gnil.

GEN sd_ulong(const char *v, long flag, const char *s, ulong *ptn, ulong Min, ulong Max, const char **msg)

• ptn points to the current numerical value of the toggle, and is set to the new value (when v is nonempty).

• Min and Max point to the minimum and maximum values allowed for the default.

• v must translate to an integer in the allowed ranger, a suffix among k/K ($\times 10^3$), m/M ($\times 10^6$), or g/G ($\times 10^9$) is allowed, but no arithmetic expression.

• msg is a [NULL]-terminated array of messages or NULL (ignored). If msg is not NULL, msg[i] contains a message attached to the value i of the default. The last entry in the msg array is used as a message attached to all subsequent ones.

The exact behavior and return value depends on flag:

• d_RETURN: returns the new value, as a GEN.

• `d_ACKNOWLEDGE`: prints a message indicating the new value, possibly a message attached to it via the `msg` argument, and return `gnil`.

• other cases: print nothing and return `gnil`.

`GEN sd_intarray(const char *v, long flag, const char *s, GEN *pz)`

• records a `t_VECSMALL` array of nonnegative integers.

• `pz` points to the current `t_VECSMALL` value, and is set to the new value (when `v` is nonempty).

The exact return value depends on `flag`:

• `d_RETURN`: returns the new value, as a `t_VEC` (converted via `zv_to_ZV`)

• `d_ACKNOWLEDGE`: prints a message indicating the new value, (as a `t_VEC`) and return `gnil`.

• other cases: print nothing and return `gnil`.

`GEN sd_string(const char *v, long flag, const char *s, char **pstr)` • `v` is subjet to environment expansion, then time expansion.

• `pstr` points to the current string value, and is set to the new value (when `v` is nonempty).

12.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

`GEN obj_init(long d, long n)` returns an *obj* S, a `t_VEC` with d regular components, accessed as `gel(S,1)`, ..., `gel(S,d)`; together with a record of n members, all initialized to 0. The arguments d and n must be nonnegative.

After `S = obj_init(d, n)`, the prototype of our other functions are of the form

```
GEN obj_do(GEN S, long tag, ...)
```

The first argument S holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to n) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if $n = 3$, after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

Technical note. In the current implementation, S is a `t_VEC` with $d+1$ entries. The first d components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S,d+1)`, a `t_VEC` of length n initialized to `zerovec`(n), must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate garbage collections. This mechanism allows to simulate C `structs`, in a simpler way than with general hashtables, while remaining compatible with the GP language (which knows neither structs nor hashtables). It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gc_upto`), rather than having to deal with individual components of actual C `structs`.

`GEN obj_reinit(GEN S)` make a shallow copy of S, re-initializing all dynamic components. This allows "forking" a lazy vector while avoiding both a memory leak, and storing pointers to the same data in different objects (with risks of a double free later).

`GEN obj_check(GEN S, long tag)` if the *tag*-component in S is non empty, return it. Otherwise return `NULL`. The `t_INT` 0 (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of) O as *tag*-component of S. Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting O as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of S is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* has relative p-adic precision $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_realprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* satisfies `gprecision` $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN, long), GEN (*gpr)(GEN), long prec)` if the *tag*-component of S is non empty *and* has precision `gpr`$(x) \geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`void obj_free(GEN S)` destroys all clones stored in the n tagged components, and replace them by the initial value 0. The regular entries of S are unaffected, and S remains a valid object. This is used to avoid memory leaks.

Chapter 13:
Algebraic Number Theory

13.1 General Number Fields.

13.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the **Z**-module generated by its columns may not be an $\mathbf{Z}_K$-module (i.e. the expensive `nfisideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a `bnf` structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to `NULL` (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to `NULL` (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`GEN checkbnf_i(GEN bnf)` same as `checkbnf` but return `NULL` instead of raising an exception.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`GEN checkbnr_i(GEN bnr)` same as `checkbnr` but returns the *bnr* or `NULL` instead of raising an exception.

`GEN checknf_i(GEN nf)` same as `checknf` but return `NULL` instead of raising an exception.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`int checkrnf_i(GEN rnf)` same as `checkrnf` but return 0 on failure and 1 on success.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

`GEN checkbid_i(GEN bid)` same as `checkbid` but return `NULL` instead of raising an exception and return `bid` on success.

`GEN checkznstar_i(GEN G)` return G if it is a *znstar*; else return `NULL` on failure.

`GEN checkgal(GEN x)` if a *galoisinit* structure can be extracted from `x`, return it; otherwise raise an exception.

`void checksqmat(GEN x, long N)` check whether `x` is a square matrix of dimension `N`. May be used to check for ideals if `N` is the field degree.

`void checkprid(GEN pr)` Raise an exception if the argument is not a prime ideal structure.

`int checkprid_i(GEN pr)` same as `checkprid` but return 0 instead of raising an exception and return 1 on success.

`int is_nf_factor(GEN F)` return 1 if F is an ideal factorization and 0 otherwise.

`int is_nf_extfactor(GEN F)` return 1 if F is an extended ideal factorization (allowing 0 or negative exponents) and 0 otherwise.

`int RgV_is_prV(GEN v)` returns 1 if the vector v contains only prime ideals and 0 otherwise.

`GEN get_prid(GEN ideal)` return the underlying prime ideal structure if one can be extracted from `ideal` (ideal or extended ideal), and return `NULL` otherwise.

`void checkabgrp(GEN v)` Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries: $[N, cyc]$ or $[N, cyc, gen]$.

`GEN abgrp_get_no(GEN x)` extract the cardinality N from an abelian group structure.

`GEN abgrp_get_cyc(GEN x)` extract the elementary divisors *cyc* from an abelian group structure.

`GEN abgrp_get_gen(GEN x)` extract the generators *gen* from an abelian group structure.

`GEN cyc_get_expo(GEN cyc)` return the exponent of the group with structure `cyc`; 0 for an infinite group.

`void checkmodpr(GEN modpr)` Raise an exception if the argument is not a `modpr` structure (from `nfmodprinit`).

`GEN get_modpr(GEN x)` return x if it is a `modpr` structure and `NULL` otherwise.

`GEN checknfelt_mod(GEN nf, GEN x, const char *s)` given an *nf* structure `nf` and a `t_POLMOD` `x`, return the attached polynomial representative (shallow) if `x` and `nf` are compatible. Raise an exception otherwise. Set s to the name of the caller for a meaningful error message.

`int check_ZKmodule_i(GEN x)` return 1 if x looks like a projective $\mathbf{Z}_K$-module, i.e., a pair $[A, I]$ where A is a matrix and I is a list of ideals and A has as many columns as I has elements. Or possibly a longer list $[A, I, \ldots]$ such as the output of `rnfpseudobasis`. Otherwise return 0.

`void check_ZKmodule(GEN x, const char *s)` raise an exception unless x is recognized as a projective $\mathbf{Z}_K$-module. Set s to the name of the caller for a meaningful error message.

`long idealtyp(GEN *ideal, GEN *fa)` The input is `ideal`, a pointer to an ideal or extended ideal; returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

As a first side effect, `*ideal` is set to the underlying ideal, possibly simplified (for instance the zero ideal represented by an empty matrix is replaced by `gen_0`).

If `fa` is not `NULL`, then `*fa` is set to the extended part in the input: either `NULL` (regular ideal) or the extended part of an extended ideal.

13.1.2 Extracting info from a `nf` structure.

These functions expect a true *nf* argument attached to a number field $K = \mathbf{Q}[x]/(T)$, e.g. a *bnf* will not work. Let $n = [K : \mathbf{Q}]$ be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial T (monic, in $\mathbf{Z}[x]$).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places r_1.

`long nf_get_r2(GEN nf)` returns the number of complex places r_2.

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets r_1 and r_2 to the number of real and complex places respectively. Note that $r_1 + 2r_2$ is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree, $n = r_1 + 2r_2$.

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of T, i.e. the index of the order generated by the power basis $(1, x, \dots, x^{n-1})$ in the maximal order of K.

`GEN nf_get_zk(GEN nf)` returns a basis $(w_1, w_2, \dots, w_n)$ for the maximal order of K. Those are polynomials in $\mathbf{Q}[x]$ of degree $< n$; it is guaranteed that $w_1 = 1$.

`GEN nf_get_zkden(GEN nf)` returns the denominator of `nf_get_zk`, as a positive `t_INT`.

`GEN nf_get_zkprimpart(GEN nf)` returns `nf_get_zk` times its denominator.

`GEN nf_get_invzk(GEN nf)` returns the matrix $(m_{i,j}) \in M_n(\mathbf{Z})$ giving the power basis (x^i) in terms of the (w_j), i.e such that $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$ for all $1 \le j \le n$; since $w_1 = 1 = x^0$, we have $m_{i,1} = \delta_{i,1}$ for all i. The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the r_1 real roots of the polynomial defining the number fields: first the r_1 real roots (as `t_REAL`s), then the r_2 representatives of the pairs of complex conjugates.

`GEN ZX_nf_roots(GEN T, long r1, long prec)` computes the same r_1+r_2 roots as `nf_get_roots` from scratch, starting with a squarefree `ZX` T, assuming it has exactly r_1 real roots. Note that `QX_complex_roots` returns all $r_1 + 2r_2$ complex roots.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of T: first the r_1 real roots (as `t_REAL`s), then the r_2 pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K: $M[i, j]$ contains $w_j(\alpha_i)$, where α_i is the i-th element of `nf_get_roots(nf)`. In particular, if v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i]w_i$ of K, then `RgM_RgC_mul(M,v)` represents the embeddings of v.

`GEN nf_get_G(GEN nf)` returns a $n \times n$ real matrix G such that $Gv \cdot Gv = T_2(v)$, where v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i]w_i$ of K and T_2 is the standard Euclidean form on $K \otimes \mathbf{R}$, i.e. $T_2(v) = \sum_\sigma |\sigma(v)|^2$, where σ runs through all n complex embeddings of K.

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of G, rounded to nearest integers, specifically `RM_round_maxrank`(G).

`GEN nf_get_ramified_primes(GEN nf)` returns the vector of ramified primes.

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis $(w_1, \dots, w_n)$: its (i, j) entry is $\mathrm{Tr} w_i w_j$.

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

13.1.3 Extracting info from a `bnf` structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

`GEN bnf_get_nf(GEN bnf)` returns the underlying *nf*.

`GEN bnf_get_clgp(GEN bnf)` returns the class group in *bnf*, which is a 3-component vector $[h, cyc, gen]$.

`GEN bnf_get_cyc(GEN bnf)` returns the elementary divisors of the class group (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

`GEN bnf_get_gen(GEN bnf)` returns the generators $[g_1, \dots, g_k]$ of the class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$.

`GEN bnf_get_no(GEN bnf)` returns the class number.

`GEN bnf_get_reg(GEN bnf)` returns the regulator.

`GEN bnf_get_logfu(GEN bnf)` returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

`GEN bnf_get_fu(GEN bnf)` returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

`GEN bnf_get_fu_nocheck(GEN bnf)` as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

`GEN bnf_get_tuU(GEN bnf)` returns a generator of the torsion part of $\mathbf{Z}_K^*$.

`long bnf_get_tuN(GEN bnf)` returns the order of the torsion part of $\mathbf{Z}_K^*$, i.e. the number of roots of unity in K.

`GEN bnf_get_sunits(GEN bnf)` allows access to the algebraic data stored by `bnfinit(,1)`. The function returns `NULL` unless the `bnf` was initialized by `bnfinit(,1)`, else a vector $[X, U, E, \mathtt{lim}]$ where

• X is a vector of rational primes and algebraic integers all of whose prime divisors have norm less than `lim`,

• U is a matrix of exponents whose columns yield the fundamental units `bnf.fu`. More precisely,

$$\mathtt{bnf.fu}[j] = \prod_i X[i]^{U[i,j]}.$$

• E is a matrix of exponents whose columns yield the generators of principal ideals attached to the HNF of the `bnf` relation matrix between the maximal ideals of norm less `lim` (that generate the class group under GRH). More precisely, `bnf[5]` contains the prime factor base P (its first r elements being independant class group generators), `bnf[1]` contains a matrix W in HNF in $M_r(\mathbf{Z})$ and `bnf[2]`, contains a matrix B in $M_{r\times c}(\mathbf{Z})$. We define algebraic numbers e_j for $j \le r + c$ such that

$$\prod_{\mathtt{i}\le\mathtt{r}} \mathtt{P}_\mathtt{i}^{\mathtt{W[i,j]}} = (e_j), \quad j \le r$$

$$P_j \prod_{\mathtt{i}\le\mathtt{r}} \mathtt{P}_\mathtt{i}^{\mathtt{B[i,j]}} = (e_j), j > r$$

Then $e_j = \prod_i X[i]^{E[i,j]}$.

`GEN bnf_has_fu(GEN bnf)` return fundamental units in expanded form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu(GEN bnf)` return fundamental units as a vector of algebraic numbers in compact form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu_mat(GEN bnf)` as a pair (X, U), where X is a vector of S-units and U is a matrix with integer entries (without 0 rows), see `bnf_get_sunits`, if `bnf` contains them. Else return `NULL`.

13.1.4 Extracting info from a `bnr` structure.

These functions expect a true *bnr* argument.

`GEN bnr_get_bnf(GEN bnr)` returns the underlying *bnf*.

`GEN bnr_get_nf(GEN bnr)` returns the underlying *nf*.

`GEN bnr_get_clgp(GEN bnr)` returns the ray class group.

`GEN bnr_get_no(GEN bnr)` returns the ray class number.

`GEN bnr_get_cyc(GEN bnr)` returns the elementary divisors of the ray class group (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bnr_get_gen(GEN bnr)` returns the generators $[g_1, \ldots, g_k]$ of the ray class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$. Raise a generic error if the *bnr* does not contain the ray class group generators.

`GEN bnr_get_gen_nocheck(GEN bnr)` as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

`GEN bnr_get_bid(GEN bnr)` returns the *bid* attached to the *bnr* modulus.

`GEN bnr_get_mod(GEN bnr)` returns the modulus attached to the *bnr*.

13.1.5 Extracting info from an `rnf` structure.

These functions expect a true *rnf* argument, attached to an extension L/K, $K = \mathbf{Q}[y]/(T)$, $L = K[x]/(P)$.

`long rnf_get_degree(GEN rnf)` returns the *relative* degree $[L : K]$.

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree $[L : \mathbf{Q}]$.

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field $[K : \mathbf{Q}]$.

`GEN rnf_get_nf(GEN rnf)` returns the base field K, an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial T defining the base field K.

`long rnf_get_nfvarn(GEN rnf)` returns the variable y attached to the base field K.

`GEN rnf_get_nfzk(GEN rnf)` returns the integer basis of the base field K.

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining L/K.

`long rnf_get_varn(GEN rnf)` returns the variable x attached to L.

`GEN rnf_get_zk(GEN nf)` returns the relative integer basis generating $\mathbf{Z}_L$ as a $\mathbf{Z}_K$-module, as a pseudo-matrix (A, I) in HNF.

`GEN rnf_get_disc(GEN rnf)` is the output $[\mathfrak{d}, s]$ of `rnfdisc`.

`GEN rnf_get_ramified_primes(GEN rnf)` returns the vector of rational primes below ramified primes in the relative extension, i.e. all prime numbers appearing in the factorization of

```
idealnorm(rnf_get_nf(rnf), rnf_get_disc(rnf));
```

`GEN rnf_get_idealdisc(GEN rnf)` is the ideal discriminant $\mathfrak{d}$ from `rnfdisc`.

`GEN rnf_get_index(GEN rnf)` is the index ideal $\mathfrak{f}$

`GEN rnf_get_polabs(GEN rnf)` returns an absolute polynomial defining $L/\mathbf{Q}$.

`GEN rnf_get_alpha(GEN rnf)` a root α of the polynomial defining the base field, modulo `polabs` (cf. `rnfequation`)

`GEN rnf_get_k(GEN rnf)` a small integer k such that $\theta = \beta + k\alpha$ is a root of `polabs`, where β is a root of `pol` and α a root of the polynomial defining the base field, as in `rnf_get_alpha` (cf. also `rnfequation`).

`GEN rnf_get_invzk(GEN rnf)` contains A^{-1}, where (A, I) is the chosen pseudo-basis for $\mathbf{Z}_L$ over $\mathbf{Z}_K$.

`GEN rnf_get_map(GEN rnf)` returns technical data attached to the map $K \to L$. Currently, this contains data from `rnfequation`, as well as the polynomials T and P.

13.1.6 Extracting info from a `bid` structure.

These functions expect a true *bid* argument, attached to a modulus $I = I_0 I_\infty$ in a number field K. The underlying abelian group is $G = (\mathbf{Z}_K/I)^*$. Not that if the *bid* was initialized by `Idealstarmod` with a non-`NULL` `cycmod` argument, computations take place in $G/G^{\texttt{cycmod}}$ instead.

`GEN bid_get_mod(GEN bid)` returns the modulus attached to the *bid*.

`GEN bid_get_MOD(GEN bid)` returns the integer `cycmod` given as argument to `Idealstarmod`, or `NULL` if we used `Idealstar` to initialize `bid`.

`GEN bid_get_grp(GEN bid)` returns the abelian group attached to $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_ideal(GEN bid)` return the finite part I_0 of the *bid* modulus (an integer ideal).

`GEN bid_get_arch(GEN bid)` return the Archimedean part I_∞ of the *bid* modulus as a vector of real places in vec01 format, see Section 13.1.20.

`GEN bid_get_archp(GEN bid)` return the Archimedean part I_∞ of the *bid* modulus, as a vector of real places in indices format see Section 13.1.20.

`GEN bid_get_fact(GEN bid)` returns the ideal factorization $I_0 = \prod_i \mathfrak{p}_i^{e_i}$.

`GEN bid_get_fact2(GEN bid)` as `bid_get_fact` with all factors $\mathfrak{p}^1$ with $\mathfrak{p}$ of norm 2 removed from the factorization. (They play no role in the structure of $(\mathbf{Z}_K/I)^*$, except that the generators must be made coprime to them.)

`bid_get_ideal`(*bid*), via `idealfactor`.

`GEN bid_get_no(GEN bid)` returns the cardinality of the group $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_cyc(GEN bid)` returns the elementary divisors of the group $(\mathbf{Z}_K/I)^*$ (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bid_get_gen(GEN bid)` returns the generators of $(\mathbf{Z}_K/I)^*$ contained in *bid*. Raise a generic error if *bid* does not contain generators.

`GEN bid_get_gen_nocheck(GEN bid)` as `bid_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bid* yourself!

`GEN bid_get_sprk(GEN bid)` return a list of structures attached to the $(\mathbf{Z}_K/\mathfrak{p}^e)^*$ where $\mathfrak{p}^e$ divides I_0 exactly.

`GEN bid_get_sarch(GEN bid)` return the structure attached to $(\mathbf{Z}_K/I_\infty)^*$, by `nfarchstar`.

`GEN bid_get_U(GEN bid)` return the matrix with integral coefficients relating the local generators (from chinese remainders) to the global SNF generators (*bid*`.gen`).

13.1.7 Extracting info from a `znstar` structure.

These functions expect an argument G as returned by `znstar0(N, 1)`, attached to a positive N and the abelian group $(\mathbf{Z}/N\mathbf{Z})^*$. Let (g_i) be the SNF generators, where g_i has order d_i; we call (g'_i) the (canonical) Conrey generators, where g'_i has order d'_i. Both sets of generators have the same cardinality.

`GEN znstar_get_N(GEN bid)` return N.

`GEN znstar_get_faN(GEN G)` return the factorization `factor`(N), $N = \prod_j p_j^{e_j}$.

`GEN znstar_get_pe(GEN G)` return the vector of primary factors $(p_j^{e_j})$.

`GEN znstar_get_no(GEN G)` the cardinality $\phi(N)$ of G.

`GEN znstar_get_cyc(GEN G)` elementary divisors (d_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

`GEN znstar_get_gen(GEN G)` SNF generators divisors (g_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

`GEN znstar_get_conreycyc(GEN G)` orders (d'_i) of Conrey generators.

`GEN znstar_get_conreygen(GEN G)` Conrey generators (g'_i).

`GEN znstar_get_U(GEN G)` a square matrix U such that $(g_i) = U(g'_i)$.

`GEN znstar_get_Ui(GEN G)` a square matrix U' such that $U'(g_i) = (g'_i)$. In general, UU' will not be the identity.

13.1.8 Inserting info in a number field structure.

If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

These functions expect a `bnf` argument:

`GEN bnf_build_cycgen(GEN bnf)` the *bnf* contains generators $[g_1, \dots, g_k]$ of the class group, each with order d_i. Then $g_i^{d_i} = (x_i)$ is a principal ideal. This function returns the x_i as a factorization matrix (`famat`) giving the element in factored form as a product of S-units.

`GEN bnf_build_matalpha(GEN bnf)` the class group was computed using a factorbase S of prime ideals $\mathfrak{p}_i$, $i \leq r$. They satisfy relations of the form $\prod_j \mathfrak{p}_i^{e_{i,j}} = (\alpha_j)$, where the $e_{i,j}$ are given by the matrices $bnf[1]$ (W, singling out a minimal set of generators in S) and $bnf[2]$ (B, expressing the rest of S in terms of the singled out generators). This function returns the α_j in factored form as a product of S-units.

`GEN bnf_build_units(GEN bnf)` returns a minimal set of generators for the unit group in expanded form. The first element is a torsion unit, the others have infinite order. This expands units in compact form contained in a `bnf` from `bnfinit`$(, 1)$ and may be *very* expensive if the units are huge.

`GEN bnf_build_cheapfu(GEN bnf)` as `bnf_build_units` but only expand units in compact form if the computation is inexpensive (a few seconds). Return `NULL` otherwise.

These functions expect a `rnf` argument:

`GEN rnf_build_nfabs(GEN rnf, long prec)` given a *rnf* structure attached to L/K, (compute and) return an *nf* structure attached to L at precision `prec`.

`void rnfcomplete(GEN rnf)` as `rnf_build_nfabs` using the precision of K for `prec`.

`GEN rnf_zkabs(GEN rnf)` returns a $\mathbf{Z}$-basis in HNF for $\mathbf{Z}_L$ as a pair $[T, v]$, where T is `rnf_get_polabs`(rnf) and v a vector of elements lifted from $\mathbf{Q}[X]/(T)$. Note that the function `rnf_build_nfabs` essentially applies `nfinit` to the output of this function.

13.1.9 Increasing accuracy.

`GEN nfnewprec(GEN x, long prec)`. Raise an exception if `x` is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to `prec` and return the new structure. This is mostly useful with `prec` larger than the accuracy to which `x` was computed, but it is also possible to decrease the accuracy of `x` (truncating relevant components, which may speed up later computations). This routine may modify the original `x` (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to `bnfisprincipal0`, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original* `x` so that later precision increases become very fast. In particular, the copy returned by `nfnewprec` also contains this additional data.

`GEN bnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnf* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN bnrnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnr* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN rnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *rnf* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN nfnewprec_shallow(GEN nf, long prec)`

`GEN bnfnewprec_shallow(GEN bnf, long prec)`

`GEN rnfnewprec_shallow(GEN rnf, long prec)`

`GEN bnrnewprec_shallow(GEN bnr, long prec)` Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.

13.1.10 Number field arithmetic. The number field $K = \mathbf{Q}[X]/(T)$ is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to K is given as

- a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo T), or
- a `t_POLMOD` (modulo T), or
- a `t_COL v` of dimension $N = [K : \mathbf{Q}]$, representing the element in terms of the computed integral basis (e_i), as

```
sum(i = 1, N, v[i] * nf.zk[i])
```

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

Safe routines. The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

`GEN nfadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN nfsub(GEN nf, GEN x, GEN y)` returns $x - y$.

`GEN nfdiv(GEN nf, GEN x, GEN y)` returns x/y.

`GEN nfinv(GEN nf, GEN x)` returns x^{-1}.

`GEN nfmul(GEN nf, GEN x, GEN y)` returns xy.

`GEN nfpow(GEN nf, GEN x, GEN k)` returns x^k, k is in $\mathbf{Z}$.

`GEN nfpow_u(GEN nf, GEN x, ulong k)` returns x^k, $k \geq 0$; the argument `nf` is a true *nf* structure.

`GEN nfsqr(GEN nf, GEN x)` returns x^2.

`long nfval(GEN nf, GEN x, GEN pr)` returns the valuation of x at the maximal ideal `p` attached to the *prid* `pr`. Returns `LONG_MAX` if x is 0.

`GEN nfnorm(GEN nf, GEN x)` absolute norm of x.

`GEN nftrace(GEN nf, GEN x)` absolute trace of x.

`GEN nfpoleval(GEN nf, GEN pol, GEN a)` evaluate the `t_POL pol` (with coefficients in `nf`) on the algebraic number a (also in nf).

`GEN FpX_FpC_nfpoleval(GEN nf, GEN pol, GEN a, GEN p)` evaluate the `FpX pol` on the algebraic number a (also in nf).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

`GEN nfdiveuc(GEN nf, GEN a, GEN b)` returns the algebraic integer closest to x/y. Functionally identical to `ground( nfdiv(nf,x,y) )`.

`GEN nfdivrem(GEN nf, GEN a, GEN b)` returns the vector $[q, r]$, where

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));    \\ or r = nfmod(nf,a,b);
```

`GEN nfmod(GEN nf, GEN a, GEN b)` returns r such that

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));
```

`GEN nf_to_scalar_or_basis(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfalgtobasis`). Shallow function.

`GEN nf_to_scalar_or_alg(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (lifted `nfbasistoalg`). Shallow function.

`GEN nf_to_scalar_or_polmod(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its `t_POLMOD` representation (`nfbasistoalg`). Shallow function.

`GEN nfV_to_scalar_or_alg(GEN nf, GEN v)` aplly `nf_to_scalar_or_alg` to all components of vector v.

`GEN RgX_to_nfX(GEN nf, GEN x)` let x be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

`GEN RgM_to_nfM(GEN nf, GEN x)` let x be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

`GEN RgC_to_nfC(GEN nf, GEN x)` let x be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

`GEN nfX_to_monic(GEN nf, GEN T, GEN *pL)` given a nonzero `t_POL` T with coefficients in nf, return a monic polynomial f with integral coefficients such that $f(x) = CT(x/L)$ for some integral L and some C in nf. The function allows coefficients in basis form; if $L \neq 1$, it will return them in algebraic form. If `pL` is not `NULL`, `*pL` is set to L. Shallow function.

Unsafe routines. The following routines assume that their `nf` argument is a true nf (e.g. a bnf is not allowed) and their argument are restricted in various ways, see the precise description below.

`GEN nfX_disc(GEN nf, GEN A)` given an nf structure attached to a number field K with main variable Y (`nf_get_varn(nf)`), a `t_POL` $A \in K[X]$ given as a lift in $\mathbf{Q}[X,Y]$ (implicitly modulo `nf_get_pol(nf)`, return the discriminant of A as a `t_POL` in $\mathbf{Q}[Y]$ (representing an element of K).

`GEN nfX_resultant(GEN nf, GEN A, GEN B)` analogous to `nfX_disc`, $A, B \in \mathbf{Q}[X,Y]$; return the resultant of A and B with respect to X as a `t_POL` in $\mathbf{Q}[Y]$ (representing an element of K).

`GEN nfinvmodideal(GEN nf, GEN x, GEN A)` given an algebraic integer x and a nonzero integral ideal A in HNF, returns a y such that $xy \equiv 1$ modulo A.

`GEN nfpowmodideal(GEN nf, GEN x, GEN n, GEN ideal)` given an algebraic integer x, an integer n, and a nonzero integral ideal A in HNF, returns an algebraic integer congruent to x^n modulo A.

`GEN nfmuli(GEN nf, GEN x, GEN y)` returns $x \times y$ assuming that both x and y are either `t_INT`s or `ZV`s of the correct dimension. The argument `nf` is a true nf structure.

`GEN nfsqri(GEN nf, GEN x)` returns x^2 assuming that x is a `t_INT` or a `ZV` of the correct dimension. The argument `nf` is a true nf structure.

`GEN nfC_nf_mul(GEN nf, GEN v, GEN x)` given a `t_VEC` or `t_COL` v of elements of K in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element x (arbitrary form). This is faster than multiplying coordinatewise since pre-computations related to x (computing the multiplication table) are done only once. The components of the result are in most cases `t_COL`s but are allowed to be `t_INT`s or `t_FRAC`s. Shallow function.

`GEN nfC_multable_mul(GEN v, GEN mx)` same as `nfC_nf_mul`, where the argument x is replaced by its multiplication table `mx`.

`GEN zkC_multable_mul(GEN v, GEN x)` same as `nfC_nf_mul`, where v is a vector of algebraic integers, x is an algebraic integer, and x is replaced by `zk_multable`(x).

`GEN zk_multable(GEN nf, GEN x)` given a `ZC` x (implicitly representing an algebraic integer), returns the `ZM` giving the multiplication table by x. Shallow function (the first column of the result points to the same data as x).

`GEN zk_inv(GEN nf, GEN x)` given a `ZC` x (implicitly representing an algebraic integer), returns the `QC` giving the inverse x^{-1}. Return `NULL` if x is 0. Not stack-clean but safe for `gc_upto`.

`GEN zkmultable_inv(GEN mx)` as `zk_inv`, where the argument given is `zk_multable`(x).

`GEN zkmultable_capZ(GEN mx)` given a nonzero *zkmultable* mx attached to $x \in \mathbf{Z}_K$, return the positive generator of $(x) \cap \mathbf{Z}$.

`GEN zk_scalar_or_multable(GEN nf, GEN x)` given a `t_INT` or `ZC` x, returns a `t_INT` equal to x if the latter is a scalar (`t_INT` or `ZV_isscalar`(x) is 1) and `zk_multable`(nf, x) otherwise. Shallow function.

13.1.11 Number field arithmetic for linear algebra.

The following routines implement multiplication in a commutative R-algebra, generated by $(e_1 = 1, \dots, e_n)$, and given by a multiplication table M: elements in the algebra are n-dimensional `t_COL`s, and the matrix M is such that for all $1 \leq i, j \leq n$, its column with index $(i-1)n + j$, say (c_k), gives $e_i \cdot e_j = \sum c_k e_k$. It is assumed that e_1 is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If x has any other type than `t_COL` where an algebra element is expected, it is understood as xe_1.

`GEN multable(GEN M, GEN x)` given a column vector x, representing the quantity $\sum_{i=1}^{N} x_i e_i$, returns the multiplication table by x. Shallow function.

`GEN ei_multable(GEN M, long i)` returns the multiplication table by the i-th basis element e_i. Shallow function.

`GEN tablemul(GEN M, GEN x, GEN y)` returns $x \cdot y$.

`GEN tablesqr(GEN M, GEN x)` returns x^2.

`GEN tablemul_ei(GEN M, GEN x, long i)` returns $x \cdot e_i$.

`GEN tablemul_ei_ej(GEN M, long i, long j)` returns $e_i \cdot e_j$.

`GEN tablemulvec(GEN M, GEN x, GEN v)` given a vector v of elements in the algebra, returns the $x \cdot v[i]$.

The following routines implement naive linear algebra using the *black box field* mechanism:

`GEN nfM_det(GEN nf, GEN M)`

`GEN nfM_inv(GEN nf, GEN M)`

`GEN nfM_ker(GEN nf, GEN M)`

`GEN nfM_mul(GEN nf, GEN A, GEN B)`

`GEN nfM_nfC_mul(GEN nf, GEN A, GEN B)`

13.1.12 Cyclotomic field arithmetic for linear algebra.

The following routines implement modular algorithms in cyclotomic fields. In the prototypes, P is the n-th cyclotomic polynomial Φ_n and M is a `t_MAT` with `t_INT` or `ZX` coefficients, understood modulo P.

`GEN ZabM_ker(GEN M, GEN P, long n)` returns an integral (primitive) basis of the kernel of M.

`GEN ZabM_indexrank(GEN M, GEN P, long n)` return a vector with two `t_VECSMALL` components givin the rank profile of M. Inefficient (but correct) when M does not have almost full column rank.

`GEN ZabM_inv(GEN M, GEN P, long n, GEN *pden)` assume that M is invertible; return N and sets the algebraic integer `*pden` (an integer or a `ZX`, implicitly modulo P) such that $MN = \texttt{den}\cdot\mathrm{Id}$.

`GEN ZabM_pseudoinv(GEN M, GEN P, long n, GEN *pv, GEN *pden)` analog of `ZM_pseudoinv`. Not stack-clean.

`GEN ZabM_inv_ratlift(GEN M, GEN P, long n, GEN *pden)` return a primitive matrix H such that MH is d times the identity and set `*pden` to d. Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use `ZabM_inv`.

13.1.13 Cyclotomic trace.

Given two positive integers m and n such that $K_m = \mathbf{Q}(\zeta_m) \subset K_n = \mathbf{Q}(\zeta_n)$, these functions implement relative trace computation from K_n to K_m. This is in particular useful for character values.

`GEN Qab_trace_init(long n, long m, GEN Pn, GEN Pm)` assume that `Pn` is `polcyclo`(n), `Pm` is `polcyclo`(m) (both in the same variable), initialize a structure T used in the following routines. Shallow function.

`GEN Qab_tracerel(GEN T, long t, GEN z)` assume T was created by `Qab_trace_init`, t is an integer such that $0 \le t < [K_n : K_m]$ and z belongs to the cyclotomic field $\mathbf{Q}(\zeta_n) = \mathbf{Q}[X]/(\texttt{Pn})$. Return the normalized relative trace $[K_n : K_m]^{-1}\mathrm{Tr}_{K_n/K_m}(\zeta_n^t z)$. Shallow function.

`GEN QabV_tracerel(GEN T, long t, GEN v)` v being a vector of entries belonging to K_n, apply `Qab_tracerel` to all entries. Shallow function.

`GEN QabM_tracerel(GEN T, long t, GEN m)` m being a matrix of entries belonging to K_n, apply `Qab_tracerel` to all entries. Shallow function.

13.1.14 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on $\mathbf{Z}$-modules with a natural multiplicative structure (K^*, fractional ideals in K, $\mathbf{Z}_K^*$, ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like $C_i \leftarrow C_i - 10000C_1$ involves raising entries in C_1 to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INT`s, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns g and e as one meaningful object when fully expanded as $\prod g[i]^{e[i]}$, *famat*s are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famat*s just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object x which is not of type `t_MAT` will be treated as the factorization x^1. The following functions all return *famat*s:

`GEN famat_mul(GEN f, GEN g)` f, g are *famat*, or objects whose type is *not* `t_MAT` (understood as f^1 or g^1). Returns fg. The empty factorization is the neutral element for *famat* multiplication.

`GEN famat_mul_shallow(GEN f, GEN g)` shallow version of `famat_mul`.

`GEN famat_pow(GEN f, GEN n)` n is a `t_INT`. If f is a `t_MAT`, assume it is a *famat* and return f^n (multiplies the exponent column by n). Otherwise, understand it as an element and returns the 1-line *famat* f^n.

`GEN famat_pow_shallow(GEN f, GEN n)` shallow version of `famat_pow`.

`GEN famat_pows_shallow(GEN f, long n)` shallow version of `famat_pow` where n is a small integer.

`GEN famat_mulpow_shallow(GEN f, GEN g, GEN e)` `famat` corresponding to $f \cdot g^e$. Shallow function.

`GEN famat_mulpows_shallow(GEN f, GEN g, long e)` `famat` shallow version of `famat_mulpow` where e is a small integer.

`GEN famat_sqr(GEN f)` returns f^2.

`GEN famat_inv(GEN f)` returns f^{-1}.

`GEN famat_div(GEN f, GEN g)` return f/g.

`GEN famat_inv_shallow(GEN f)` shallow version of `famat_inv`.

`GEN famat_div_shallow(GEN f, GEN g)` return f/g; shallow.

`GEN famat_Z_gcd(GEN M, GEN n)` restrict the *famat* M to the prime powers dividing n.

`GEN to_famat(GEN x, GEN k)` given an element x and an exponent k, returns the *famat* x^k.

`GEN to_famat_shallow(GEN x, GEN k)` same, as a shallow function.

`GEN Z_to_famat(GEN x)` converts the `t_INT` x to a *famat*. This does not factor x but will replace it by y^k with y integral and k maximal. Note that 0 gets converted to 0^1. Shallow function.

`GEN Q_to_famat(GEN x)` converts the `t_INT` or `t_FRAC` x to a *famat*. If x is a `t_INT`, as `Z_to_famat`. Else, this does not factor x but will replace it by $y^a z^{-b}$ with integral y and z and maximal (positive) a and b. Shallow function.

`GEN famatV_factorback(GEN v, GEN e)` given a vector of `famat`s v and a `ZV` e return the `famat` $\prod_i v[i]^{e[i]}$. Shallow function.

`GEN famatV_zv_factorback(GEN v, GEN e)` given a vector of `famat`s v and a `zv` e return the `famat` $\prod_i v[i]^{e[i]}$. Shallow function.

`GEN ZM_famat_limit(GEN f, GEN limit)` given a *famat* f with `t_INT` entries, returns a *famat* g with all factors larger than `limit` multiplied out as the last entry (with exponent 1). Shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COL`s of the same length.

`GEN famat_reduce(GEN f)` given a *famat* f, returns a *famat* g without repeated elements or 0 exponents, such that the expanded forms of f and g would be equal. Shallow function.

`GEN famat_remove_trivial(GEN f)` given a *famat* f, returns a *famat* g without 0 exponents. Shallow function.

`GEN famatsmall_reduce(GEN f)` as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

`GEN famat_to_nf(GEN nf, GEN f)` You normally never want to do this! This is a simplified form of `nffactorback`, where we do not check the user input for consistency. The elements must be regular algebraic numbers (not *famat*s) over the given number field.

Why should you *not* want to use this function ? You should not need to: most of the functions useful in this context accept *famat*s as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can hopefully make good use of a quotient operation (modulo a fixed conductor, modulo ℓ-th powers); see the end of Section 13.1.26. If nothing else works, this function is available but is expected to be slow or even overflow the possibilities of the implementation.

`GEN famat_idealfactor(GEN nf, GEN x)` This is a good alternative for `famat_to_nf`, returning the factorization of the ideal generated by x. Since the answer is still given in factorized form, there is no risk of coefficient explosion when the exponents are large. Of course, all components of x must be factored individually.

`GEN famat_nfvalrem(GEN nf, GEN x, GEN pr, GEN *py)` return the valuation v at `pr` of `famat_to_nf`(x), without performing the expansion of course. Notive that the output is a `GEN` since it cannot be assumed to fit into a `long`. If `py` is not `NULL` it contains the `famat` obtained by applying `nfvalrem` to each entry of the first column and copying the second column, with 0 exponents removed. The expanded algebraic number is coprime to `pr` (in fact, all its components are coprime do `pr`) and equal to $x\tau^v$ where τ is the fixed anti-uniformizer for `pr` (`pr_get_tau`).

Caveat. Receiving a *famat* input, `bnfisunit` assumes that it is an actual unit, since this is expensive to check, and normally easy to ensure from the user's side.

13.1.15 Ideal arithmetic.

Conversion to HNF.

`GEN idealhnf(GEN nf, GEN x)` where the argument `nf` is a true *nf* structure. Returns the HNF of the ideal defined by x: x may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

• if strictly less than $N = [K : Q]$ generators are given, x is the $\mathbf{Z}_K$-module they generate,

• if N or more are given, it is assumed that they form a $\mathbf{Z}$-basis (that the matrix has maximal rank N). This acts as `mathnf` since the $\mathbf{Z}_K$-module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

`GEN idealhnf0(GEN nf, GEN x, GEN y)` returns the HNF of the ideal generated by the two algebraic numbers x and y.

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

`GEN idealhnf_principal(GEN nf, GEN x)` returns the ideal generated by the algebraic number x.

`GEN idealhnf_shallow(GEN nf, GEN x)` a shallow `idealhnf`: if x is already in HNF, we return x, not a copy!

`GEN idealhnf_two(GEN nf, GEN v)` assuming $a = v[1]$ is a nonzero `t_INT` and $b = v[2]$ is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by b, see `zk_multable`), returns the HNF of $a\mathbf{Z}_K + b\mathbf{Z}_K$.

`GEN nfV_idealhnf(GEN nf, GEN v, GEN *pden)` v being a vector of elements of nf with common denominator d over the integral basis, returns the HNF of the ideal generated by the elements of dv and set $*pden = d$ if $d \neq 1$, otherwise set $*pden$ to `NULL`.

Operations.

The basic ideal routines accept all `nf`s (*nf*, *bnf*, *bnr*) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

`GEN idealadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN idealdiv(GEN nf, GEN x, GEN y)` returns x/y. Returns an extended ideal if x or y is an extended ideal.

`GEN idealmul(GEN nf, GEN x, GEN y)` returns xy. Returns an extended ideal if x or y is an extended ideal.

`GEN idealsqr(GEN nf, GEN x)` returns x^2. Returns an extended ideal if x is an extended ideal.

`GEN idealinv(GEN nf, GEN x)` returns x^{-1}. Returns an extended ideal if x is an extended ideal.

`GEN idealpow(GEN nf, GEN x, GEN n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealpows(GEN nf, GEN ideal, long n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealmulred(GEN nf, GEN x, GEN y)` returns an extended ideal equal to xy.

`GEN idealpowred(GEN nf, GEN x, GEN n)` returns an extended ideal equal to x^n.

More specialized routines suffer from various restrictions:

`GEN idealdivexact(GEN nf, GEN x, GEN y)` returns x/y, assuming that the quotient is an integral ideal. Much faster than `idealdiv` when the norm of the quotient is small compared to Nx. Strips the principal parts if either x or y is an extended ideal.

`GEN idealdivpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns $x\mathfrak{p}^{-n}$, assuming x is an ideal in HNF or a rational number, and `pr` a *prid* attached to $\mathfrak{p}$. Shallow function: it returns x when $n = 0$. The `nf` argument must be a true *nf* structure.

`GEN idealmulpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns $x\mathfrak{p}^n$, assuming x is an ideal in HNF or a rational number, and `pr` a *prid* attached to $\mathfrak{p}$. Shallow function: it returns x when $n = 0$. The `nf` argument must be a true *nf* structure.

`GEN idealprodprime(GEN nf, GEN v)` given a list v of prime ideals in *prid* form, return their product. Assume that *nf* is a true *nf* structure.

`GEN idealprod(GEN nf, GEN v)` given a list v of ideals, return their product.

`GEN idealprodval(GEN nf, GEN v, GEN pr)` given a list v of ideals return the valuation of their product at the prime ideal `pr`.

`GEN idealHNF_mul(GEN nf, GEN x, GEN y)` returns xy, assuming than `nf` is a true *nf*, x is an integral ideal in HNF and y is an integral ideal in HNF or precompiled form (see below). For maximal speed, the second ideal y may be given in precompiled form $y = [a, b]$, where a is a nonzero `t_INT` and b is an algebraic integer in regular representation (a `t_MAT` giving the multiplication table by the fixed element): very useful when many ideals x are going to be multiplied by the same ideal y. This essentially reduces each ideal multiplication to an $N \times N$ matrix multiplication followed by a $N \times 2N$ modular HNF reduction (modulo $xy \cap \mathbf{Z}$).

`GEN idealHNF_inv(GEN nf, GEN I)` returns I^{-1}, assuming that `nf` is a true *nf* and x is a fractional ideal in HNF.

`GEN idealHNF_inv_Z(GEN nf, GEN I)` returns $(I \cap \mathbf{Z}) \cdot I^{-1}$, assuming that `nf` is a true *nf* and x is an integral fractional ideal in HNF. The result is an integral ideal in HNF.

`GEN ideals_by_norm(GEN nf, GEN N)` given a true *nf* structure and a integer N, which can also be given by a factorization matrix or (preferably) by a pair $[N, \texttt{factor}(N)]$, return all ideals of norm N in factored form. Not stack-clean.

Approximation.

`GEN idealaddtoone(GEN nf, GEN A, GEN B)` given to coprime integer ideals A, B, returns $[a, b]$ with $a \in A$, $b \in B$, such that $a + b = 1$ The result is reduced mod AB, so a, b will be small.

`GEN idealaddtoone_i(GEN nf, GEN A, GEN B)` as `idealaddtoone` except that `nf` must be a true *nf*, and only a is returned.

`GEN idealaddtoone_raw(GEN nf, GEN A, GEN B)` as `idealaddtoone_i` except that the reduction mod AB is only performed modulo the lcm of $A \cap \mathbf{Z}$ and $B \cap \mathbf{Z}$, which will increase the size of a.

`GEN zkchineseinit(GEN nf, GEN A, GEN B, GEN AB)` given two coprime integral ideals A and B (in any form, preferably HNF) and their product `AB` (in HNF form), initialize a solution to the Chinese remainder problem modulo AB. THe `nf` argument must be a true *nf* structure.

`GEN zkchinese(GEN zkc, GEN x, GEN y)` given `zkc` from `zkchineseinit`, and x, y two integral elements given as `t_INT` or `ZC`, return a z modulo AB such that $z = x \bmod A$ and $z = y \bmod B$.

`GEN zkchinese1(GEN zkc, GEN x)` as `zkchinese` for $y = 1$; useful to lift elements in a nice way from $(\mathbf{Z}_K/A_i)^*$ to $(\mathbf{Z}_K/\prod_i A_i)^*$.

`GEN hnfmerge_get_1(GEN A, GEN B)` given two square upper HNF integral matrices A, B of the same dimension $n > 0$, return a in the image of A such that $1 - a$ is in the image of B. (By abuse of notation we denote 1 the column vector $[1, 0, \ldots, 0]$.) If such an a does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

`GEN idealaddmultoone(GEN nf, GEN v)` given a list of n (globally) coprime integer ideals $(v[i])$ returns an n-dimensional vector a such that $a[i] \in v[i]$ and $\sum a[i] = 1$. If $[K : \mathbf{Q}] = N$, this routine computes the HNF reduction (with $Gl_{nN}(\mathbf{Z})$ base change) of an $N \times nN$ matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

`GEN idealapprfact(GEN nf, GEN fx)` as `idealappr`, except that x *must* be given in factored form. (This is unchecked.)

`GEN idealcoprime(GEN nf, GEN x, GEN y)`. Given 2 integral ideals x and y, returns an algebraic number α such that αx is an integral ideal coprime to y.

`GEN idealcoprimefact(GEN nf, GEN x, GEN fy)` same as `idealcoprime`, except that y is given in factored form, as from `idealfactor`.

`GEN idealchinese(GEN nf, GEN x, GEN y)`

`GEN idealchineseinit(GEN nf, GEN x)`

13.1.16 Maximal ideals.

The PARI structure attached to maximal ideals is a *prid* (for *pr*ime *id*eal), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* attached to a maximal ideal $\mathfrak{p}$ stores the following data: the underlying rational prime p, the ramification degree $e \geq 1$, the residue field degree $f \geq 1$, a p-uniformizer π with valuation 1 at $\mathfrak{p}$ and valuation 0 at all other primes dividing p and a rescaled "anti-uniformizer" τ used to compute valuations. This τ is an algebraic integer such that τ/p has valuation -1 at $\mathfrak{p}$ and is integral at all other primes; in particular, the valuation of $x \in \mathbf{Z}_K$ is positive if and only if the algebraic integer $x\tau$ is divisible by p (easy to check for elements in `t_COL` form).

`GEN pr_get_p(GEN pr)` returns p. Shallow function.

`GEN pr_get_gen(GEN pr)` returns π. Shallow function.

`long pr_get_e(GEN pr)` returns e.

`long pr_get_f(GEN pr)` returns f.

`GEN pr_get_tau(GEN pr)` returns `zk_scalar_or_multable`(nf, τ), which is the `t_INT` 1 iff p is inert, and a `ZM` otherwise. Shallow function.

`int pr_is_inert(GEN pr)` returns 1 if p is inert, 0 otherwise.

`GEN pr_norm(GEN pr)` returns the norm p^f of the maximal ideal.

`ulong upr_norm(GEN pr)` returns the norm p^f of the maximal ideal, as an `ulong`. Assume that the result does not overflow.

`GEN pr_hnf(GEN pr)` return the HNF of $\mathfrak{p}$.

`GEN pr_inv(GEN pr)` return the fractional ideal $\mathfrak{p}^{-1}$, in HNF.

`GEN pr_inv_p(GEN pr)` return the integral ideal $p\mathfrak{p}^{-1}$, in HNF.

`GEN idealprimedec(GEN nf, GEN p)` list of maximal ideals dividing the prime p.

`GEN idealprimedec_limit_f(GEN nf, GEN p, long f)` as `idealprimedec`, limiting the list to primes of residual degree $\leq f$ if f is nonzero.

`GEN idealprimedec_limit_norm(GEN nf, GEN p, GEN B)` as `idealprimedec`, limiting the list to primes of norm $\leq B$, which must be a positive `t_INT`.

`int prime_over_p_full(GEN P)` given a vector of distinct primes above the same rational prime p, return 1 if it contains all primes above p.

`GEN idealprimedec_galois(GEN nf, GEN p)` return a single prime ideal above p. The `nf` argument is a true nf structure.

`GEN idealprimedec_degrees(GEN nf, GEN p)` return a (sorted) `t_VECSMALL` containing the residue degrees $f(\mathfrak{p}/p)$. The `nf` argument is a true nf structure.

`GEN idealprimedec_kummer(GEN nf, GEN Ti, long ei, GEN p)` let nf (true nf) correspond to $K = \mathbf{Q}[X]/(T)$ (T monic `ZX`). Let $T \equiv \prod_i T_i^{e_i} \pmod p$ be the factorization of T and let (f, g, h) be as in Dedekind criterion for prime p: $f \equiv \prod T_i$, $g \equiv \prod T_i^{e_i - 1}$, $h = (T - fg)/p$, and let D be the gcd of (f, g, h) in $\mathbf{F}_p[X]$. Let `Ti` (`FpX`) be one irreducible factor T_i not dividing D, with `ei` $= e_i$. This function returns the prime ideal attached to T_i by Kummer / Dedekind criterion, namely $p\mathbf{Z}_K + T_i(\bar{X})\mathbf{Z}_K$, which has ramification index e_i over p. The `nf` argument is a true nf structure. Shallow function.

`GEN idealfactor(GEN nf, GEN x)` factors the fractional (hence nonzero) ideal x into prime ideal powers; return the factorization matrix.

`GEN idealfactor_limit(GEN nf, GEN x, ulong lim)` as `idealfactor`, including only prime ideals above rational primes $<$ `lim`.

`GEN idealfactor_partial(GEN nf, GEN x, GEN L)` return partial factorization of fractional ideal x as limited by argument L:

- $L =$ `NULL`: as `idealfactor`;
- L a `t_INT`: as `idealfactor_limit`;
- L a vector of prime ideals of nf and/or rational primes (standing for "all prime ideal divisors of given rational prime") limit factorization to trial division by elements of L; do not include the cofactor. For efficiency, the list should not contain the same element twice, nor both a rational prime and one of its prime ideal divisors, but the function will work in that case as well.

`GEN idealHNF_Z_factor(GEN x, GEN *pvN, GEN *pvZ)` given an integral (nonzero) ideal x in HNF, compute both the factorization of Nx and of $x \cap \mathbf{Z}$. This returns the vector of prime divisors of both and sets `*pvN` and `*pvZ` to the corresponding `t_VECSMALL` vector of exponents for the factorization for the Norm and intersection with $\mathbf{Z}$ respetively.

`GEN idealHNF_Z_factor_i(GEN x, GEN fa, GEN *pvN, GEN *pvZ)` internal variant of `idealHNF_Z_factor` where `fa` is either a partial factorization of $x \cap \mathbf{Z}$ ($= x[1, 1]$) or `NULL`. Returns the prime divisors of x above the rational primes in `fa` and attached `vN` and `vZ`. If `fa` is `NULL`, use the full factorization, i.e. identical to `idealHNF_Z_factor`.

`GEN nf_pV_to_prV(GEN)`nf, GEN P given a vector of rational primes P, return the vector of all prime ideals above the $P[i]$.

`GEN nf_deg1_prime(GEN nf)` let `nf` be a true nf. This function returns a degree 1 (unramified) prime ideal not dividing `nf.index`. In fact it returns an ideal above the smallest prime $p \geq [K : \mathbf{Q}]$ satisfying those conditions.

`GEN prV_lcm_capZ(GEN L)` given a vector L of *prid* (maximal ideals) return the squarefree positive integer generating their lcm intersected with $\mathbf{Z}$. Not stack-clean.

`GEN prV_primes(GEN)`GEN L given a vector of `prid`, return the (sorted) list of rational primes P they divide. Not stack-clean but suitable for `gc_upto`.

GEN pr_uniformizer(GEN pr, GEN F) given a *prid* attached to $\mathfrak{p}/p$ and F in $\mathbf{Z}$ divisible exactly by p, return an F-uniformizer for pr, i.e. a t in $\mathbf{Z}_K$ such that $v_{\mathfrak{p}}(t) = 1$ and $(t, F/\mathfrak{p}) = 1$. Not stack-clean.

GEN pr_anti_uniformizer(GEN nf, GEN pr) given a number field nf and a *prid* attached to a prime ideal $\mathfrak{p}/p$ of ramification index e, return an anti-uniformizer for pr, i.e. a t in $\mathbf{Z}_K$ such that $t = 1 \bmod \mathfrak{p}$ and $t = 0 \bmod p\mathfrak{p}^{-e}$. Not stack-clean.

13.1.17 Decomposition groups.

GEN idealramfrobenius(GEN nf, GEN gal, GEN pr, GEN ram) Let K be the number field defined by nf and assume $K/\mathbf{Q}$ be a Galois extension with Galois group given gal=galoisinit(nf), and that pr is the prime ideal $\mathfrak{P}$ in prid format, and that $\mathfrak{P}$ is ramified, and ram is its list of ramification groups as output by idealramgroups. This function returns a permutation of gal.group which defines an automorphism σ in the decomposition group of $\mathfrak{P}$ such that if p is the unique prime number in $\mathfrak{P}$, then $\sigma(x) \equiv x^p \bmod \mathbf{P}$ for all $x \in \mathbf{Z}_K$.

GEN idealramfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN ram, GEN aut) as idealramfrobenius(nf, gal, pr, ram.

GEN idealramgroups_aut(GEN nf, GEN gal, GEN pr, GEN aut) as idealramgroups(nf, gal, pr.

GEN idealfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN aut) faster version of idealfrobenius(nf, gal, pr where aut must be equal to nfgaloispermtobasis(nf, gal).

13.1.18 Reducing modulo maximal ideals.

GEN nfmodprinit(GEN nf, GEN pr) returns an abstract modpr structure, attached to reduction modulo the maximal ideal pr, in idealprimedec format. From this data we can quickly project any pr-integral number field element to the residue field.

GEN modpr_get_pr(GEN x) return the pr component from a modpr structure.

GEN modpr_get_p(GEN x) return the p component from a modpr structure (underlying rational prime).

GEN modpr_get_T(GEN x) return the T component from a modpr structure: either NULL (prime of degree 1) or an irreducible FpX defining the residue field over $\mathbf{F}_p$.

In library mode, it is often easier to use directly

GEN nf_to_Fq_init(GEN nf, GEN *ppr, GEN *pT, GEN *pp) concrete version of nfmodprinit: nf and *ppr are the inputs, the return value is a modpr and *ppr, *pT and *pp are set as side effects.

The input *ppr is either a maximal ideal or already a modpr (in which case it is replaced by the underlying maximal ideal). The residue field is realized as $\mathbf{F}_p[X]/(T)$ for some monic $T \in \mathbf{F}_p[X]$, and we set *pT to T and *pp to p. Set $T =$ NULL if the prime has degree 1 and the residue field is $\mathbf{F}_p$.

In short, this receives (or initializes) a modpr structure, and extracts from it T, p and $\mathfrak{p}$.

GEN nf_to_Fq(GEN nf, GEN x, GEN modpr) returns an Fq congruent to x modulo the maximal ideal attached to modpr. The output is canonical: all elements in a given residue class are represented by the same Fq.

`GEN Fq_to_nf(GEN x, GEN modpr)` returns an `nf` element lifting the residue field element x, either a `t_INT` or an algebraic integer in `algtobasis` format.

`GEN modpr_genFq(GEN modpr)` Returns an `nf` element whose image by `nf_to_Fq` is $X \pmod T$, if $\deg T > 1$, else 1.

`GEN zkmodprinit(GEN nf, GEN pr)` as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an x whose rational denominator is not 0 in the residue field (i.e. if the valuation of x is nonnegative at all primes dividing p).

`GEN zk_to_Fq_init(GEN nf, GEN *pr, GEN *T, GEN *p)` as `nf_to_Fq_init`, able to reduce only p-integral elements.

`GEN zk_to_Fq(GEN x, GEN modpr)` as `nf_to_Fq`, for a p-integral x.

`GEN nfM_to_FqM(GEN M, GEN nf, GEN modpr)` reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

`GEN FqM_to_nfM(GEN M, GEN modpr)` lifts an `FqM` to a matrix of `nf` elements.

`GEN nfV_to_FqV(GEN A, GEN nf, GEN modpr)` reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as `A` (`t_VEC` or `t_COL`).

`GEN FqV_to_nfV(GEN A, GEN modpr)` lifts an `FqV` to a vector of `nf` elements (same type as `A`).

`GEN nfX_to_FqX(GEN Q, GEN nf, GEN modpr)` reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

`GEN FqX_to_nfX(GEN Q, GEN modpr)` lifts an `FqX` to a polynomial with coefficients in `nf`.

The following functions are technical and avoid computing a true `nfmodpr`:

`GEN pr_basis_perm(GEN nf, GEN pr)` given a true nf structure and a prime ideal `pr` above p, return as a `t_VECSMALL` the $f(\mathfrak{p}/p)$ indices i such that the `nf.zk`$[i]$ mod p form an $\mathbf{F}_p$-basis of the residue field.

`GEN QXQV_to_FpM(GEN v, GEN T, GEN p)` let p be a positive integer, v be a vector of n polynomials with rational coefficients whose denominators are coprime to p, and T be a `ZX` (preferably monic) of degree d whose leading coefficient is coprime to p. Return the $d \times n$ `FpM` whose columns are the $v[i]$ mod T, p in the canonical basis $1, X, \dots, X^{d-1}$, see `RgX_to_RgC`. This is for instance useful when v contains a $\mathbf{Z}$-basis of the maximal order of a number field $\mathbf{Q}[X]/(P)$, p is a prime not dividing the index of P and T is an irreducible factor of P mod p, attached to a maximal ideal $\mathfrak{p}$: left-multiplication by the matrix maps number field elements (in basis form) to the residue field of $\mathfrak{p}$.

`GEN hnf_Znproj(GEN H)` let H be a $n \times n$ HNF matrix which is an identity apart from the first row and assume that $q = H[1,1]$ is positive. Then $\mathbf{Z}^n/\mathrm{im}_{\mathbf{Z}} H \simeq \mathbf{Z}/q\mathbf{Z}$ and this function returns a vector v allowing to map any $x \in \mathbf{Z}^n$ to the quotient: the image is `FpV_dotproduct`(x, v, q).

The conditions are for instance satisfied when H is the HNF matrix attached to a prime ideal of degree 1 (because we assume that the implied fixed $\mathbf{Z}$-basis for the maximal order starts with 1). This allows to map algebraic integers to the (prime) residue field. This function is a special case of `zkmodprinit`.

13.1.19 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return $v_P(x)$

Unsafe functions. assume that P, Q are `prid`.

`long ZC_nfval(GEN x, GEN P)` returns $v_P(x)$, assuming x is a `ZC`, representing a nonzero algebraic integer.

`long ZC_nfvalrem(GEN x, GEN P, GEN *newx)` returns $v = v_P(x)$, assuming x is a `ZC`, representing a nonzero algebraic integer, and sets `*newx` to $x\tau^v$ which is an algebraic integer coprime to p.

`int ZC_prdvd(GEN x, GEN P)` returns 1 is P divides x and 0 otherwise. Assumes that x is a `ZC`, representing an algebraic integer. Faster than computing $v_P(x)$.

`int pr_equal(GEN P, GEN Q)` returns 1 is P and Q represent the same maximal ideal: they must lie above the same p and share the same e, f invariants, but the p-uniformizer and τ element may differ. Returns 0 otherwise.

13.1.20 Signatures.

"Signs" of the real embeddings of number field element are represented in additive notation, using the standard identification $(\mathbf{Z}/2\mathbf{Z}, +) \to (\{-1, 1\}, \times)$, $s \mapsto (-1)^s$.

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under GP, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (GP: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC` v with `t_INT` entries return as a `t_VECSMALL` the list of indices i such that $v[i] \neq 0$. (Typically used with 0, 1-vectors but not necessarily so.) If v is already a `t_VECSMALL`, return it: not stack-clean in this case.

`GEN vecsmall01_to_indices(GEN v)` as

```
vec01_to_indices(zv_to_ZV(v));
```

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length n with ones exactly at the positions $p[1], p[2], \ldots$

`GEN nfsign(GEN nf, GEN x)` x being a number field element and `nf` any form of number field, return the $0 - 1$-vector giving the signs of the r_1 real embeddings of x, as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications. The argument `nf` is a true *nf* structure.

If x is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the $x[i]$.

`GEN nfsign_arch(GEN nf, GEN x, GEN arch)` `arch` being a list of distinct real places, either in `vec01` (`t_VEC` with `gen_0` or `gen_1` entries) or `indices` (`t_VECSMALL`) form (see `vec01_to_indices`), returns the signs of x at the corresponding places. This is the low-level function underlying `nfsign`. The argument `nf` is a true *nf* structure.

`int nfchecksigns(GEN nf, GEN x, GEN pl)` *pl* is a `t_VECSMALL` with r_1 components, all of which are in $\{-1, 0, 1\}$. Return 1 if $\sigma_i(x)pl[i] \geq 0$ for all i, and 0 otherwise.

`GEN nfsign_units(GEN bnf, GEN archp, int add_tu)` `archp` being a divisor at infinity in `indices` form (or `NULL` for the divisor including all real places), return the signs at `archp` of a

`bnf.tu` and of system of fundamental units for the field `bnf.fu`, in that order if `add_tu` is set; and in the same order as `bnf.fu` otherwise.

`GEN nfsign_fu(GEN bnf, GEN archp)` returns the signs at `archp` of the fundamental units `bnf.fu`. This is an alias for `nfsign_units` with `add_tu` unset.

`GEN nfsign_tu(GEN bnf, GEN archp)` returns the signs at `archp` of the torsion unit generator `bnf.tu`.

`GEN nfsign_from_logarch(GEN L, GEN invpi, GEN archp)` given L the vector of the $\log\sigma(x)$, where σ runs through the (real or complex) embeddings of some number field, `invpi` being a floating point approximation to $1/\pi$, and `archp` being a divisor at infinity in `indices` form, return the signs of x at the corresponding places. This is the low-level function underlying `nfsign_units`; the latter is actually a trivial wrapper `bnf` structures include the $\log\sigma(x)$ for a system of fundamental units of the field.

`GEN set_sign_mod_divisor(GEN nf, GEN x, GEN y, GEN sarch)` let $f = f_0 f_\infty$ be a divisor, let `sarch` be the output of `nfarchstar(nf, f0, finf)`, let x encode a vector of signs at the places of f_∞ (see below), and let y be a nonzero number field element. Returns z congruent to y mod f_0 (integral if y is) such that z and x have the same signs at f_∞. The argument `nf` is a true *nf* structure.

The following formats are supported for x: a $\{0,1\}$-vector of signs as a `t_VECSMALL` (0 for positive, 1 for negative); `NULL` for a totally positive element (only 0s); a number field element which is replaced by its signature at f_∞.

`GEN nfarchstar(GEN nf, GEN f0, GEN finf)` for a divisor $f = f_0 f_\infty$ represented by the integral ideal `f0` in HNF and the `finf` in `indices` form, returns $(\mathbf{Z}_K/f_\infty)^*$ in a form suitable for computations mod f. See `set_sign_mod_divisor`.

`GEN idealprincipalunits(GEN nf, GEN pr, long e)` returns the multiplicative group $(1+pr)/(1+pr^e)$ as an abelian group. Faster than `idealstar` when the norm of pr is large, since it avoids (useless) work in the multiplicative group of the residue field.

13.1.21 Complex embeddings.

`GEN nfembed(GEN nf, GEN x, long k)` returns a floating point approximation of the k-th embedding of x (attached to the k-th complex root in `nf.roots`). Note that the semantic is different from `nfeltembed` (which increases the precision of *nf* until the embeddings have the requested precision): `nfembed` provides the embedding with the precision that is achievable given *nf*.

`GEN nfeltembed_i(GEN *pnf, GEN x, GEN ind, long prec)` as `nfeltembed`, except that no garbage collecting is performed and `*pnf` must be initially set to a (true) *nf* structure and, if the routine needs to increase the accuracy of *nf* to achieve the requested accuracy `prec`, then the functions sets it to the new more precise *nf*.

`GEN nf_cxlog(GEN nf, GEN x, long prec)` return the vector of complex logarithmic embeddings $(e_i \mathrm{Log}(\sigma_i X))$ where $e_i = 1$ if $i \le r_1$ and $e_i = 2$ if $r_1 < i \le r_2$ of $X =$ `Q_primpart`(x). Returns `NULL` if loss of accuracy. Not stack-clean but suitable for `gc_upto`. Allows x in compact representation, in which case `Q_primpart` is taken componentwise.

`GEN nf_cxlog_normalize(GEN nf, GEN x, long prec)` an *nf* structure attached to a number field K and x from `nf_cxlog`(nf, X) (a column vector of complex logarithmic embeddings with

$r_1 + r_2$ components) and let $e = (e_1, \dots, e_{r_1+r_2})$. Return

$$x - \frac{\log\left(N_{K/\mathbf{Q}}X\right)}{[K:\mathbf{Q}]}e$$

where the imaginary parts are further normalized modulo $2\pi i \cdot e$.

The composition `nf_cxlog` followed by `nf_cxlog_normalize` is a morphism from $(K^*/\mathbf{Q}_+^*, \times)$ to $((\mathbf{C}/2\pi i\mathbf{Z})^{r_1} \times (\mathbf{C}/4\pi i\mathbf{Z})^{r_2}, +)$. Its real part maps the units $\mathbf{Z}_K^*$ to a lattice in the hyperplane $\sum_i x_i = 0$ in $\mathbf{R}^{r_1+r_2}$.

`GEN nfV_cxlog(GEN nf, GEN x, long prec)` applies `nf_cxlog` to each component of the vector x. Returns `NULL` if loss of accuracy for even one component. Not stack-clean.

`GEN nflogembed(GEN nf, GEN x, GEN *emb, long prec)` return the vector of real logarithmic embeddings $(e_i \mathrm{Log}|\sigma_i x|)$ where $e_i = 1$ if $i \le r_1$ and $e_i = 2$ if $r_1 < i \le r_2$. Returns `NULL` if loss of accuracy. Not stack-clean. If `emb` is non-`NULL` set it to $(e_i\sigma_i x)$. Allows x in compact representation, in which case `emb` is returned in compact representation as well, as a factorization matrix (expanding the factorization may overflowexponents).

13.1.22 Maximal order and discriminant, conversion to `nf` structure.

A number field $K = \mathbf{Q}[X]/(T)$ is defined by a monic $T \in \mathbf{Z}[X]$. The low-level function computing a maximal order is

`void nfmaxord(nfmaxord_t *S, GEN T0, long flag)`, where the polynomial T_0 is squarefree with integer coefficients. Let K be the étale algebra $\mathbf{Q}[X]/(T_0)$ and let $T =$ `ZX_Q_normalize`(T_0), i.e. $T = CT_0(X/L)$ is monic and integral for some $C, Q \in \mathbf{Q}$.

The structure `nfmaxord_t` is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of Q[X]/(T) */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to $T = T_0$, i.e. either to pass the input polynomial through `ZX_Q_normalize` *before* the call, or to forget about T_0 and go on with the polynomial T; otherwise `unscale` $\neq 1$, all data is expressed in terms of $T \neq T_0$, and needs to be converted to T_0. For instance to convert the basis to $\mathbf{Q}[X]/(T_0)$:

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing T (monic `ZX`), one can use the format $[T, listP]$ as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags, both of them deprecated:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `factorlimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime $p >$ `factorlimit` such that p^2 divides the field discriminant `dK`. This flag is *deprecated*: the $[T, listP]$ format is safer and more flexible.

`nf_ROUND2`: this flag is *deprecated* and now ignored.

`void nfinit_basic(nfmaxord_t *S, GEN T0)` a wrapper around `nfmaxord` (without the deprecated `flag`) that also accepts number field structures (*nf*, *bnf*, ...) for `T0`.

`GEN nfmaxord_to_nf(nfmaxord_t *S, GEN ro, long prec)` convert an `nfmaxord_t` to an *nf* structure at precision `prec`, where `ro` is `NULL`. The argument `ro` may also be set to a vector with $r_1 + r_2$ components containing the roots of `S->T` suitably ordered, i.e. first r_1 `t_REAL` roots, then r_2 `t_COMPLEX` representing the conguate pairs, but this is *strongly discouraged*: the format is error-prone, and it is hard to compute the roots to the right accuracy in order to achieve `prec` accuracy for the `nf`. This function uses the integer basis `S->basis` as is, *without* performing LLL-reduction. Unless the basis is already known to be reduced, use rather the following higher-level function:

`GEN nfinit_complete(nfmaxord_t *S, long flag, long prec)` convert an `nfmaxord_t` to an *nf* structure at precision `prec`. The `flag` has the same meaning as in `nfinit0`. If `S->basis` is known to be reduced, it will be faster to use `nfmaxord_to_nf`.

`GEN indexpartial(GEN T, GEN dT)` T a monic separable `ZX`, `dT` is either `NULL` (no information) or a multiple of the discriminant of T. Let $K = \mathbf{Q}[X]/(T)$ and $\mathbf{Z}_K$ its maximal order. Returns a multiple of the exponent of the quotient group $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$. In other word, a *denominator* d such that $dx \in \mathbf{Z}[X]/(T)$ for all $x \in \mathbf{Z}_K$.

`GEN FpX_gcd_check(GEN x, GEN y, GEN D)` let x and y be two coprime polynomials with integer coefficients and let D be a factor of the resultant of x and y; try to factor D by running the Euclidean algorithm on x and y modulo D. This returns `NULL` or a non trivial factor of D. This is the low-level function underlying `poldiscfactors` (applied to x, `ZX_deriv`(x) and the discriminant of x). It succeeds when D has at least two prime divisors p and q such that one sub-resultant of x and y is divisible by p but not by q.

13.1.23 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

`GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. The `bnf` structure already contains information about the class group in the form $\oplus_{i=1}^{n}(\mathbf{Z}/d_i\mathbf{Z})g_i$ for canonical integers d_i (with $d_n \mid \ldots \mid d_1$ all > 1) and essentially random generators g_i, which are ideals in HNF. We normally do not need the value of the g_i, only that they are fixed once and for all and that any (nonzero) fractional ideal x can be expressed uniquely as $x = (t)\prod_{i=1}^{n} g_i^{e_i}$, where $0 \le e_i < d_i$, and (t) is some principal ideal. Computing e is straightforward, but t may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair $[e, t]$, depending on `flag`, which is an or-ed combination of the following symbolic flags:

• `nf_GEN` tries to compute t. Returns $[e, t]$, with t an empty vector if the computation failed. This flag is normally useless in nontrivial situations since the next two serve analogous purposes in more efficient ways.

• `nf_GENMAT` tries to compute t in factored form, which is much more efficient than `nf_GEN` if the class group is moderately large; imagine a small ideal $x = (t)g^{10000}$: the norm of t has 10000 as many digits as the norm of g; do we want to see it as a vector of huge meaningless integers? The idea is to compute e first, which is easy, then compute (t) as $x\prod g_i^{-e_i}$ using successive `idealmulred`, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in

factored *unexpanded* form. Returns $[e, t]$, with t an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which `bnf` was computed was ridiculously low (and then `bnfinit` should not have succeeded either). Setting/unsetting `nf_GEN` has no effect when this flag is set.

• `nf_GEN_IF_PRINCIPAL` tries to compute t *only* if the ideal is principal ($e = 0$). Returns `gen_0` if the ideal is not principal. Setting/unsetting `nf_GEN` has no effect when this flag is set, but setting/unsetting `nf_GENMAT` is possible.

• `nf_FORCE` in the above, insist on computing t, even if it requires recomputing a `bnf` from scratch. This is a last resort, and normally the accuracy of a `bnf` can be increased without trouble, but it may be that some algebraic information simply cannot be recovered from what we have: see `bnfnewprec`. It should be very rare, though.

In simple cases where you do not care about t, you may use

`GEN isprincipal(GEN bnf, GEN x)`, which is a shortcut for `bnfisprincipal0(bnf, x, 0)`.

The following low-level functions are often more useful:

`GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag)` is about the same as `bnfisprincipal0` applied to $C \prod L[i]^{f[i]}$, where the $L[i]$ are ideals, the $f[i]$ integers and C is either an ideal or `NULL` (omitted). Make sure to include `nf_GENMAT` in `flag`!

`GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f)` is for delicate cases, where we must be more clever than `nf_FORCE` (it is used when trying to increase the accuracy of a *bnf*, for instance). If performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute t, it just returns a `t_INT`, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that `nf_FORCE` does exactly this internally, but goes on increasing the accuracy of the `bnf`, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in `bnf` anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using `bnfnewprec` (and keep the resulting `bnf`!), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

`void bnftestprimes(GEN bnf, GEN B)` is an ingredient to certify unconditionnally a `bnf` computed assuming GRH, cf. `bnfcertify`. Running this function successfully proves that the classes of all prime ideals of norm $\leq B$ belong to the subgroup of the class group generated by the factorbase used to compute the `bnf` (equal to the class group under GRH). If the condition is not true, then (GRH is false and) the function will run forever.

If it is known that primes of norm less than B generate the class group (through variants of Minkowski's convex body or Zimmert's twin classes theorems), then the true class group is proven to be a quotient of `bnf.clgp`.

13.1.24 Floating point embeddings, the T_2 quadratic form.

We assume the *nf* is a true `nf` structure, attached to a number field K of degree n and signature (r_1, r_2). We saw that

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K, so that if v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^{n} v[i]w_i$ of K, then `RgM_RgC_mul(M,v)` represents the embeddings of v. Its first r_1 components are real numbers (`t_INT`, `t_FRAC` or `t_REAL`, usually the latter), and the last r_2 are complex numbers (usually of `t_COMPLEX`, but not necessarily for embeddings of rational numbers).

`GEN embed_T2(GEN x, long r1)` assuming x is the vector of floating point embeddings of some algebraic number v, i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v));
```

returns $T_2(v)$. If the floating point embeddings themselves are not needed, but only the values of T_2, it is more efficient to restrict to real arithmetic and use

```
gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf,v)));
```

`GEN embednorm_T2(GEN x, long r1)` analogous to `embed_T2`, applied to the `gnorm` of the floating point embeddings. Assuming that

```
x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v)) );
```

returns $T_2(v)$.

`GEN embed_roots(GEN z, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return the $r_1 + 2r_2$ roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector z of $r_1 + r_2$ complex embeddings of the algebraic number v, return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return (a floating point approximation of) the norm of v.

13.1.25 Ideal reduction, low level.

In the following routines *nf* is a true `nf`, attached to a number field K of degree n:

`GEN nf_get_Gtwist(GEN nf, GEN v)` assuming v is a `t_VECSMALL` with $r_1 + r_2$ entries, let

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i} \varepsilon_i |\sigma_i(x)|^2,$$

where as usual the σ_i are the (real and) complex embeddings and $\varepsilon_i = 1$, resp. 2, for a real, resp. complex place. This is a twisted variant of the T_2 quadratic form, the standard Euclidean form on $K \otimes \mathbf{R}$. In applications, only the relative size of the v_i will matter.

Let $G_v \in M_n(\mathbf{R})$ be a square matrix such that if $x \in K$ is represented by the column vector X in terms of the fixed $\mathbf{Z}$-basis of $\mathbf{Z}_K$ in *nf*, then

$$||x||_v^2 = {}^t(G_v X) \cdot G_v X.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of G_v, rounded to nearest integers, specifically `RM_round_maxrank`(G_v). Suitable for `gc_upto`, but does not collect garbage. For convenience, also allow v =`NULL` (`nf_get_roundG`) and v a `t_MAT` as output from the function itself: in both these cases, shallow function.

`GEN nf_get_Gtwist1(GEN nf, long i)`. Simple special case. Returns the twisted G matrix attached to the vector v whose entries are all 0 except the i-th one, which is equal to 10.

`GEN idealpseudomin(GEN x, GEN G)`. Let x, G be two `ZM`s, such that the product Gx is well-defined. This returns a "small" integral linear combinations of the columns of x, given by the LLL-algorithm applied to the lattice Gx. Suitable for `gc_upto`, but does not collect garbage.

In applications, x is an integral ideal, G approximates a Cholesky form for the T_2 quadratic form as returned by `nf_get_Gtwist`, and we return a small element a in the lattice (x, T_2). This is used to implement `idealred`.

`GEN idealpseudomin_nonscalar(GEN x, GEN G)`. As `idealpseudomin`, but we insist of returning a nonscalar a (`ZV_isscalar` is false), if the dimension of x is > 1.

In the interpretation where x defines an integral ideal on a fixed $\mathbf{Z}_K$ basis whose first element is 1, this means that a is not rational.

`GEN idealpseudominvec(GEN x, GEN G)`. As `idealpseudomin_nonscalar`, but we return about $n^2/2$ nonscalar elements in x with small T_2-norm, where the dimension of x is n.

`GEN idealpseudored(GEN x, GEN G)`. As `idealpseudomin` but we return the full reduced $\mathbf{Z}$-basis of x as a `t_MAT` instead of a single vector.

`GEN idealred_elt(GEN nf, GEN x)` shortcut for

```
idealpseudomin(x, nf_get_roundG(nf))
```

13.1.26 Ideal reduction, high level.

Given an ideal x this means finding a "simpler" ideal in the same ideal class. The public GP function is of course available

`GEN idealred0(GEN nf, GEN x, GEN v)` finds an $a \in K^*$ such that $(a)x$ is integral of small norm and returns it, as an ideal in HNF. What "small" means depends on the parameter v, see the GP description. More precisely, a is returned by `idealpseudomin`$((x_{\mathbf{Z}})x^{(} - 1), G)$ divided by $x_{\mathbf{Z}}$, where $x_{\mathbf{Z}} = (x \cap \mathbf{Z})$ and where G is `nf_get_Gtwist`(nf, v) for $v \neq$ `NULL` and `nf_get_roundG`(nf) otherwise.

Usually one sets $v =$ `NULL` to obtain an element of small T_2 norm in x:

`GEN idealred(GEN nf, GEN x)` is a shortcut for `idealred0(nf,x,NULL)`.

The function `idealred` remains complicated to use: in order not to lose information x must be an extended ideal, otherwise the value of a is lost. There is a subtlety here: the principal ideal (a) is easy to recover, but a itself is an instance of the principal ideal problem which is very difficult given only an *nf* (once a *bnf* structure is available, `bnfisprincipal0` will recover it).

`GEN idealmoddivisor(GEN bnr, GEN x)` A proof-of-concept implementation, useless in practice. If `bnr` is attached to some modulus f, returns a "small" ideal in the same class as x in the ray class group modulo f. The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in $(\mathbf{Z}_K/f)^*$.

`GEN famat_to_nf_moddivisor(GEN nf, GEN g, GEN e, GEN bid)` given a true *nf* attached to a number field K, a *bid* structure attached to a modulus f, and an algebraic number in factored form $\prod g[i]^{e[i]}$, such that $(g[i], f) = 1$ for all i, returns a small element in $\mathbf{Z}_K$ congruent to it mod f. Note that if f contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

`GEN famat_to_nf_modideal_coprime(GEN nf, GEN g, GEN e, GEN f, GEN expo)` as above except that the ideal f is now integral in HNF (no need for a full *bid*), and we pass the exponent of the group $(\mathbf{Z}_K/f)^*$ as `expo`; any multiple will also do, at the expense of efficiency. Of course if a *bid* for f is available, if is easy to extract f and the exact value of `expo` from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set `expo` to *any* positive integer, the result is correct up to `expo`-th powers, hence exact if `expo` is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take `expo`$= 2$).

`GEN nf_to_Fp_coprime(GEN nf, GEN x, GEN modpr)` this low-level function is variant of `famat_to_nf_modideal_coprime`: *nf* is a true *nf* structure, `modpr` is from `zkmodprinit` attached to a prime of degree 1 above the prime number p, and x is either a number field element or a `famat` factorization matrix. We finally assume that no component of x has a denominator p.

What to do when the $g[i]$ are not coprime to f, but only $\prod g[i]^{e[i]}$ is? Then the situation is more complicated, and we advise to solve it one prime divisor of f at a time. Let v be the valuation attached to a maximal ideal `pr`:

`GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo)` returns an element in $(\mathbf{Z}_K/\mathtt{pr}^k)^*$ congruent to the product $\prod g[i]^{e[i]}$, assumed to be globally coprime to `pr`. As above, `expo` is any positive multiple of the exponent of $(\mathbf{Z}_K/\mathtt{pr}^k)^*$, for instance $(Nv-1)p^{k-1}$, if p is the underlying rational prime. You may use other values of `expo` (see the useful trick in `famat_to_nf_modideal_coprime`).

`GEN sunits_makecoprime(GEN g, GEN pr, GEN prk)` is a specialized variant that allows to precondition a vector of $g[i]$ assumed to be integral primes or algebraic integers so that it becomes suitable for `famat_to_nf_modideal_coprime` modulo `pr`. This is in particular useful for the output of `bnf_get_sunits`.

`GEN Idealstarprk(GEN nf, GEN pr, long k, long flag)` same as `Idealstar` for $I = \mathtt{pr}^k$. The `nf` argument is a true *nf* structure.

13.1.27 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple A, B, C, where the defining set $[A, B, C]$ can have any of the following forms: [*bnr*], [*bnr*, *subgroup*], [*bnf*, *modulus*], [*bnf*, *modulus*, *subgroup*]. You can still use directly all of (`libpari`'s routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of `libpari` programming. In particular, it does not make much sense to always input a triple A, B, C because of the fringe [*bnf*, *modulus*, *subgroup*]. The first routine to call, is thus

`GEN Buchray(GEN bnf, GEN mod, long flag)` initializes a *bnr* structure from `bnf` and modulus `mod`. `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, do not return a *bnr*, only the ray class group as an abelian group). In fact, the single most useful value of `flag` is `nf_INIT` to initialize a proper *bnr*: omitting `nf_GEN` saves a lot of time and will not adversely affect any class field theoretic function; adding `nf_GEN` makes debugging easier. The flag

0 allows to compute only the ray class group structure but will gain little time; if we only need the *order* of the ray class group, then `bnrclassno` is fastest.

Now we have a proper *bnr* encoding a `bnf` and a modulus, we no longer need the [*bnf*, *modulus*] and [*bnf*, *modulus*, *subgroup*] forms, which would internally call `Buchray` anyway. Recall that a subgroup H is given by a matrix in HNF, whose column express generators of H on the fixed generators of the ray class group that stored in our *bnr*. You may also code the trivial subgroup by `NULL`. It is also allowed to replace H by a character χ of the ray class group modulo `mod`: it represents the subgroup $\mathrm{Ker}\chi$.

`GEN bnr_subgroup_check(GEN bnr, GEN H, GEN *pdeg)` given a *bnr* attached to a modulus *mod*, check whether H represents a congruence subgroup (of the ray class group modulo *mod*) and returns a normalized representation: `NULL` for the trivial subgroup, or in HNF, reduced modulo the elementary divisors of the ray class group. In particular, if H is a character of the ray class group, the returned value is the character kernel. If `pdeg` is not `NULL`, `*pdeg` is set to the degree of the attached class field: the index of H in the ray class group.

`void bnr_subgroup_sanitize(GEN *pbnr, GEN *pH)` given a *bnr* and a congruence subgroup H (a subgroup of $G = \mathrm{Cl}_f(K)$), make sanity checks and compute the subgroup conductor. Then replace the pair to match the conductor: the *bnr* has the right conductor as modulus, and the subgroup is normalized. Instead of a *bnr*, this function also accepts a *bnf* (gets replaced by the *bnr* with trivial conductor). Instead of a subgroup, the function also accepts an integer N (replaced by $\mathrm{Cl}_f(K)^N$) or a character (replaced by its kernel). The *bnr* (or rather the underlying ray class group) is computed modulo the exponent of the Galois group G/H. Not stack-clean.

`void bnr_char_sanitize(GEN *pbnr, GEN *pchi)` same idea as `bnr_subgroup_sanitize`: we are given a *bnr* and a ray class character, make sanity checks and update the data to use the conductor as modulus. The *bnr* (or rather the underlying ray class group) is computed modulo the order of the character. Not stack-clean.

`void bnr_vecchar_sanitize(GEN *pbnr, GEN *pCHI)` same idea as `bnr_char_sanitize`: we are given a *bnr* and a non empty vector of ray class characters of the *same* conductor, make sanity checks and update the data to use the conductor as modulus. to use the conductor as modulus. The *bnr* (or rather the underlying ray class group) is computed modulo the lcm of all character orders. Not stack-clean.

`void bnr_sanitize(GEN *pbnr)` sanitize its argument to make sure we get a *bnr* on output (a *bnf* gets replaced by the *bnr* with trivial conductor). This is called as a preliminary step in all the above `sanitize` functions.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`GEN bnrconductor_factored(GEN bnr, GEN H)` return a pair $[F, fa]$ where F is the conductor of H and *fa* is the factorization of the finite part of the conductor. Shallow function. The argument H goes through `bnr_subgroup_check`, so can be a congruence subgroup, a ray class character, etc.

`GEN bnrconductor_factored_arch(GEN bnr, GEN H, GEN *parch)` H must a congruence subgroup with respect to *bnr* in normalized form, as returned by `bnr_subgroup_check`. This function returns the factorization *fa* of the finite part of the conductor or `NULL` if `fa` coincides with the finite part of the *bnr* modulus. If `parch` is not `NULL`, then `*parch` is set to the conductor at infinity, or `NULL` if it coincides with the infinite places in the *bnr* modulus. In particular, the modulus of the *bnr* is the conductor of H if and only if `*parch` and `fa` are `NULL`. Shallow function.

`GEN bnrconductor_raw(GEN bnr, GEN H)` return the conductor of H. Shallow function. The argument H goes through `bnr_subgroup_check`, so can be a congruence subgroup, a ray class character, etc.

`GEN bnrtoprimitive(GEN bnr, GEN H, GEN mod)` H must a congruence subgroup with respect to *bnr* in normalized form, as returned by `bnr_subgroup_check`. This function computes the conductor F of H and returns `NULL` if it coincides with the *bnr* modulus. If not, it returns a new *bnr* structure attached to the modulus F, without generators. If `mod` is not `NULL`, the latter ray class group is computed modulo `mod`-th powers. Not stack-clean.

This is meant to be used in conjunction with `bnrmap` (or `bnrsurjection`), which can then be used to map subgroups or characters from the original *bnr* to the new one using the `abmap_XXX` functions. Beware that `mod` must be chosen accordingly, with respect to all objects which must be mapped in this way; for instance the lcm of all character orders.

`long bnrisconductor(GEN bnr, GEN H)` returns 1 is the class field defined by the subgroup H (of the ray class group mod f coded in `bnr`) has conductor f. Returns 0 otherwise.

`GEN ideallog_units(GEN bnf, GEN bid)` return the images of the units generators `bnf.tu` and `bnf.tu` in the finite abelian group $(\mathbf{Z}_K/f)^*$ attached to `bid`.

`GEN ideallog_units0(GEN bnf, GEN bid, GEN N)` let $G = (\mathbf{Z}_K/f)^*$ be the finite abelian group attached to `bid`. Return the images of the units generators `bnf.tu` and `bnf.tu` in G/G^N. If N is `NULL`, same as `ideallog_units`.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by `bnr` and H. See the description of the GP function for details. *flag* is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient) A, B, C form to the preferred *bnr*, H form for class fields. Given A, B, C as explained above (omitted entries coded by `NULL`), return the attached *bnr*, and set H to the attached subgroup. If `addgen` is 1, make sure that if the *bnr* needed to be computed, then it contains generators.

13.1.28 Abelian maps. A map $f : A \to B$ between two abelian groups of finite type is given by a triple: $[M, cyc_A, cyc_B]$, where $cyc_A = [a_1, \dots, a_m]$ and $cyc_B = [b_1, \dots, b_n]$ are the elementary divisors for A and B (see `ZM_snf`) so that $A = \oplus_{i \le m}(\mathbf{Z}/a_i\mathbf{Z})g_i$ and $B =\sim \oplus_{j \le n}(\mathbf{Z}/b_j\mathbf{Z})G_j$. The matrix M gives the image of the generators g_i in terms of the G_j: $(f(g_i))_{i \le m} = (G_j)_{j \le n} \cdot M$. The function `bnrmap` returns such a structure.

`GEN bnrsurjection(GEN BNR, GEN bnr)` `BNR` and `bnr` defined over the same field K, for moduli F and f with $f \mid F$, returns the canonical surjection $\mathrm{Cl}_K(F) \to \mathrm{Cl}_K(f)$ as an abelian map. I.e., a triple $[M, cyc_F, cyc_f]$. M gives the image of the fixed ray class group generators of `BNR` in terms of the ones in `bnr`, cyc_F and cyc_f are the cyclic structures of `BNR` and `bnr` respectively (as per `bnr_get_cyc`). Shallow function.

`GEN abmap_kernel(GEN f)` returns the kernel of the abelian map f, ans a matrix H in HNF: the subgroup is $(g_i) \cdot H$.

`GEN abmap_subgroup_image(GEN f, GEN H)` given a subgroup H of A (its generators are the $(g_i)H$); for efficiency, H should be given in canonical form, i.e., as an HNF left divisor of $diag(a_1, \dots, a_m)$. Returns the subgroup $f(H)$ of B, as an HNF left divisor of $diag(b_1, \dots, b_n)$.

`GEN abmap_char_image(GEN f, GEN chi)` given a regular (unnormalized) character `chi`, return its image by f, as a regular (unnormalized) character with respect to cyc_B.

`GEN abmap_nchar_image(GEN f, GEN nchi)` given a normalized character `nchi`, return its image by f, still as a normalized character with respect to cyc_A.

13.1.29 Grunwald–Wang theorem.

`GEN nfgwkummer(GEN nf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `nf` contains suitable roots of unity, and directly using Kummer theory to construct the extension.

`GEN bnfgwgeneric(GEN bnf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `bnf` is a `bnfinit` structure, and calling `rnfkummer` to construct the extension.

13.1.30 Relative equations, Galois conjugates.

`GEN nfissquarefree(GEN nf, GEN P)` given P a polynomial with coefficients in nf, return 1 is P is squarefree, and 0 otherwise. If is allowed (though less efficient) to replace nf by a monic `ZX` defining the field.

`GEN rnfequationall(GEN A, GEN B, long *pk, GEN *pLPRS)` A is either an nf type (corresponding to a number field K) or an irreducible `ZX` defining a number field K. B is an irreducible polynomial in $K[X]$. Returns an absolute equation C (over $\mathbf{Q}$) for the number field $K[X]/(B)$. C is the characteristic polynomial of $b+ka$ for some roots a of A and b of B, and k is a small rational integer. Set `*pk` to k.

If `pLPRS` is not `NULL` set it to $[h_0, h_1]$, $h_i \in \mathbf{Q}[X]$, where $h_0 + h_1 Y$ is the last nonconstant polynomial in the pseudo-Euclidean remainder sequence attached to $A(Y)$ and $B(X-kY)$, leading to $C = \mathrm{Res}_Y(A(Y), B(X-kY))$. In particular $a := -h_0/h_1$ is a root of A in $\mathbf{Q}[X]/(C)$, and $X-ka$ is a root of B.

`GEN nf_rnfeq(GEN A, GEN B)` wrapper around `rnfequationall` to allow mapping $K \to L$ (`eltup`) and converting elements of L between absolute and relative form (`reltoabs`, `abstorel`), *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. In fact, since A may be a `t_POL` or an nf, the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

`GEN nf_rnfeqsimple(GEN A, GEN B)` as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

`GEN eltabstorel(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltabstorel` is more robust), `nf_rnfeq` or `nf_rnfeqsimple`, return x as an element of L/K, i.e. as a `t_POLMOD` with `t_POLMOD` coefficients. Shallow function.

`GEN eltabstorel_lift(GEN rnfeq, GEN x)` same as `eltabstorel`, except that x is returned in partially lifted form, i.e. as a `t_POL` with `t_POLMOD` coefficients.

`GEN eltreltoabs(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltreltoabs` is more robust) or `nf_rnfeq`, return x in absolute form.

`GEN nf_nfzk(GEN nf, GEN rnfeq)` `rnfeq` as given by `nf_rnfeq`, `nf` a true nf structure, return a a suitable representation of `nf.zk` allowing quick computation of the map $K \to L$ by the function

`nfeltup`, *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. The computed value is the same as in `rnf_get_nfzk`. Shallow function.

`GEN nfeltup(GEN nf, GEN x, GEN zknf)` `zknf` and is initialized by `nf_nfzk` or `rnf_get_nfzk` (but in this case `rnfeltup` is more robust); `nf` is a true *nf* structure for K, returns $x \in K$ as a (lifted) element of L, in absolute form.

`GEN rnfdisc_factored(GEN nf, GEN pol, GEN *pd)` variant of `rnfdisc` returning the relative discriminant ideal *factorization*, and setting `*pd` to the discriminant as an element in $K^*/(K^*)^2$. Shallow function. The argument `nf` is a true *nf* structure.

`GEN Rg_nffix(const char *f, GEN T, GEN c, int lift)` given a `ZX` T and a "coefficient" c supposedly belonging to $\mathbf{Q}[y]/(T)$, check whether this is a the case and return a cleaned up version of c. The string f is the calling function name, used to report errors.

This means that c must be one of `t_INT`, `t_FRAC`, `t_POL` in the variable y with rational coefficients, or `t_POLMOD` modulo T which lift to a rational `t_POL` as above. The cleanup consists in the following improvements:

- `t_POL` coefficients are reduced modulo T.
- `t_POL` and `t_POLMOD` belonging to $\mathbf{Q}$ are converted to rationals, `t_INT` or `t_FRAC`.
- if `lift` is nonzero, convert `t_POLMOD` to `t_POL`, and otherwise convert `t_POL` to `t_POLMOD`s modulo T.

`GEN RgX_nffix(const char *f, GEN T, GEN P, int lift)` check whether P is a polynomials with coefficients in the number field defined by the absolute equation $T(y) = 0$, where T is a `ZX` and returns a cleaned up version of P. This checks whether P is indeed a `t_POL` with variable compatible with coefficients in $\mathbf{Q}[y]/(T)$, i.e.

```
varncmp(varn(P), varn(T)) < 0
```

and applies `Rg_nffix` to each coefficient.

`GEN RgV_nffix(const char *f, GEN T, GEN P, int lift)` as `RgX_nffix` for a vector of coefficients.

`GEN polmod_nffix(const char *f, GEN rnf, GEN x, int lift)` given a `t_POLMOD` x supposedly defining an element of *rnf*, check this and perform `Rg_nffix` cleanups.

`GEN polmod_nffix2(const char *f, GEN T, GEN P, GEN x, int lift)` as in `polmod_nffix`, where the relative extension is explicitly defined as $L = (\mathbf{Q}[y]/(T))[x]/(P)$, instead of by an `rnf` structure.

`long numberofconjugates(GEN T, long pinit)` returns a quick multiple for the number of $\mathbf{Q}$-automorphism of the (integral, monic) `t_POL` T, from modular factorizations, starting from prime `pinit` (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use `nfgaloisconj` to be sure.

`GEN nfroots_if_split(GEN *pt, GEN T)` let `*pt` point either to a number field structure or an irreducible `ZX`, defining a number field K. Given T a monic squarefree polynomial with coefficients in $\mathbf{Z}_K$, return the list of roots of `pol` in K if the polynomial splits completely, and `NULL` otherwise. In other words, this checks whether $K[X]/(T)$ is normal over K (hence Galois since T is separable by assumption).

In the case where `*pT` is a `ZX`, the function has to compute internally a conditional `nf` attached to K , whose `nf.zk` may not define the maximal order $\mathbf{Z}_K$ (see `nfroots`); `*pT` is then replaced by the conditional `nf` to avoid losing that information.

`GEN rnfabelianconjgen(GEN nf, GEN P)` *nf* being a number field structure attached to K and P being an irreducible polynomial in $K[X]$. This function returns `gen_0` if $L = K[X]/(P)$ is not abelian over K, else it returns a pair (g, o) where g is a vector of K-automorphisms of L generating the abelian group $G = \mathrm{Gal}(L/K)$ and o is a `t_VECSMALL` of the same length giving the relative orders of the g_i: $o[1]$ is the order of g_1 and for $i \geq 2$, $o[i]$ is the order of g_i in $G/(g_1, \ldots, g_{i-1})$. The length need not be minimal: the $o[i]$ need not be the elementary divisors of G.

13.1.31 Units.

`GEN nfrootsof1(GEN nf)` returns a two-component vector $[w, z]$ where w is the number of roots of unity in the number field *nf*, and z is a primitive w-th root of unity.

`GEN nfcyclotomicunits(GEN nf, GEN zu)` where `zu` is as output by `nfrootsof1(nf)`, return the vector of the cyclotomic units in `nf` expressed over the integral basis. If $\zeta = \zeta_n$ belongs to the base field (n maximal), this function returns

• (when n is not a prime power) the $\zeta^a - 1$, for all $1 \leq a < n/2$ such that $n/(a, n)$ is not a prime power and a is a strict divisor of n.

• (all n) for p prime, $v_p(n) = k > 0$, the $(z^a - 1)/(z - 1)$, where $z = \zeta^{n/p^k}$, for all $1 < a \leq (p^k - 1)/2$, $(p, a) = 1$.

These are independent modulo torsion if n is a prime power, but not necessarily so otherwise.

`GEN sunits_mod_units(GEN bnf, GEN S)` return independent generators for $U_S(K)/U(K)$.

13.1.32 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

`GEN zidealstar(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN)`

`GEN zidealstarinit(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_INIT)`

`GEN zidealstarinitgen(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN|nf_INIT)`

`GEN idealstar0(GEN nf, GEN x, long flag)` short for `idealstarmod(nf, ideal, flag, NULL)`. Use `Idealstarmod` or `Idealstar`.

`GEN bnrinit0(GEN bnf, GEN ideal, long flag)` short for `bnrinitmod(bnf,ideal,flag,NULL)`. Use `Buchray` or `Buchraymod`.

`GEN buchimag(GEN D, GEN c1, GEN c2, GEN gCO)` short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), /*ignored*/0)
```

`GEN buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec)` short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), prec)
```

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per `nf_GEN` and not `nf_GENMAT`. Don't use them:

`GEN isprincipalforce(GEN bnf, GEN x)`

GEN isprincipalgen(GEN bnf, GEN x)

GEN isprincipalgenforce(GEN bnf, GEN x)

GEN isprincipalraygen(GEN bnr, GEN x), use bnrisprincipal.

Variants on polred: use polredbest.

GEN factoredpolred(GEN x, GEN fa)

GEN factoredpolred2(GEN x, GEN fa)

GEN smallpolred(GEN x)

GEN smallpolred2(GEN x), use Polred.

GEN polred0(GEN x, long flag, GEN p)

GEN polredabs(GEN x)

GEN polredabs2(GEN x)

GEN polredabsall(GEN x, long flun)

Superseded by bnrdisclist0:

GEN discrayabslist(GEN bnf, GEN L)

GEN discrayabslistarch(GEN bnf, GEN arch, long bound)

Superseded by idealappr (*flag*is ignored)

GEN idealappr0(GEN nf, GEN x, long flag)

Superseded by bnrconductor_raw or bnrconductor_factored, bnrconductor_factored_arch:

GEN bnrconductorofchar(GEN bnr, GEN chi)

13.2 Galois extensions of Q.

This section describes the data structure output by the function galoisinit. This will be called a gal structure in the following.

13.2.1 Extracting info from a gal structure.

The functions below expect a gal structure and are shallow. See the documentation of galoisinit for the meaning of the member functions.

GEN gal_get_pol(GEN gal) returns gal.pol

GEN gal_get_p(GEN gal) returns gal.p

GEN gal_get_e(GEN gal) returns the integer e such that gal.mod==gal.p^e.

GEN gal_get_mod(GEN gal) returns gal.mod.

GEN gal_get_roots(GEN gal) returns gal.roots.

GEN gal_get_invvdm(GEN gal) gal[4].

GEN gal_get_den(GEN gal) return gal[5].

GEN gal_get_group(GEN gal) returns gal.group.

GEN gal_get_gen(GEN gal) returns gal.gen.

GEN gal_get_orders(GEN gal) returns gal.orders.

13.2.2 Miscellaneous functions.

`GEN nfgaloispermtobasis(GEN nf, GEN gal)` return the images of the field generator by the automorphisms `gal.orders` expressed on the integral basis `nf.zk`.

`GEN nfgaloismatrix(GEN nf, GEN s)` returns the `ZM` attached to the automorphism s, seen as a linear operator expressend on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x);   /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer x.

`GEN nfgaloismatrixapply(GEN nf, GEN M, GEN x)` given an automorphism M in `nfgaloismatrix` form, return the image of x under the automorphism. Variant of `galoisapply`.

13.3 Quadratic number fields and quadratic forms.

13.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the `GEN` x is a quadratic discriminant (`t_INT`, not a square, congruent to $0, 1$ modulo 4), and raise an exception otherwise. Set `*s` to the sign of x and `*mod4` to x modulo 4 (0 or 1), unless `mod4` is `NULL`.

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

13.3.2 Class number.

Given a D congruent to 0 or 1 modulo 4, let $h(D)$ denote the class number of the order of discriminant D. The function `quadclassunit` uses index calculus and computes $h(D)$ in subexponential time in $\log|D|$ but it assumes the truth of the GRH. For imaginary quadratic orders, it is also comparatively slow for *small* values, say $|D| \leq 10^{18}$. Here are some alternatives:

`GEN classno(GEN D)` corresponds to `qfbclassno(D,0)` and is only useful for $D < 0$, uses a baby-step giant-step technique and runs in time $O(D1/4)$. The result is guaranteed correct for $|D| < 2 \cdot 10^{10}$ and fastest in that range. For larger values of $|D|$, the algorithm is no longer rigorous and may give incorrect results (we know no concrete example); it also becomes relatively less interesting compared to `quadclassunit`.

`GEN classno2(GEN D)` corresponds to `qfbclassno(D,1)` and runs in time $O(D^{1/2})$; the function is provided for testing purposes only since it is never competitive.

`GEN quadclassnoF(GEN D, GEN *pd)` returns the positive integer $h(D)/h(d)$ where d is the fundamental discriminant attached to D. If `pd` is not `NULL`, set `*pd` to d.

`GEN quadclassno(GEN D)` returns $h(D)$ using Buchmann's algorithm on the order of discriminant D. If D is not fundamental, it will usually be faster to call `coredisc2_fact` and `quadclassnoF_fact` to reduce to this case first.

`long quadclassnos(long D)` returns $h(D)$ using Buchmann's algorithm on the order of discriminant D.

`ulong unegquadclassnoF(ulong x, long *pd)` returns $h(-x)/h(d)$. Set `*pd` to d.

`ulong uposquadclassnoF(ulong x, long *pd)` returns $h(x)/h(d)$. Set `*pd` to d.

`GEN quadclassnoF_fact(GEN D, GEN P, GEN E)` let D be a fundamental discriminant, and $f = \prod_i P[i]^{E[i]}$ be a positive conductor for the order of discriminant Df^2 (P is a `ZV` and E is a `ZV` or `zv`). Returns the positive integer

$$h(Df^2)/h(D) = \frac{f}{[O_D^\times : O_{Df^2}^\times]} \prod_{p|f} \left(1 - (D/p)p^{-1}\right).$$

`ulong uquadclassnoF_fact(ulong d, long s, GEN P, GEN E)` let $s = 1$ or -1 be a sign, $D = sd$ be a fundamental discriminant, and $f = \prod_i P[i]^{E[i]}$ be a positive conductor for the order of discriminant Df^2 (P and E are `t_VECSMALL`). Returns the integer

$$h(Df^2)/h(d) = \frac{f}{[O_D^\times : O_{Df^2}^\times]} \prod_{p|f} \left(1 - (D/p)p^{-1}\right).$$

`GEN hclassno(GEN d)` returns the Hurwitz-Kronecker class number $H(d)$. These play a central role in trace fomulas and are usually needed for many consecutive values of d. Thus, the function uses a cache so that later calls for *small* consecutive values of d are instantaneous, see `getcache`. Large values of d ($d > 500000$) call `quadclassunit` individually and are not memoized.

`GEN hclassnoF_fact(GEN P, GEN E, GEN D)` return $H(Df^2)/H(D)$ assuming D is a negative fundamental discriminant, where the conductor f is given in factored form: P (`ZV`) is the list of prime divisors of f and E (`t_VECSMALL`) their multiplicities.

`long uhclassnoF_fact(GEN faf, long D)` return $H(Df^2)/H(D)$ assuming D is a negative fundamental discriminant and $d = Df^2$ is an `ulong` and `faf` is `factoru`(d).

`GEN hclassno6(GEN d)` assuming $d > 0$, returns the integer $6H(d)$. This is a low-level function behind `hclassno`.

`ulong hclassno6u(ulong d)` assuming $d > 0$, returns the integer $6H(d)$. Using this function creates (or extends) caches of Hurwitz class numbers and Corediscs of negative integers to speed up consecutive or repeated calls (see `getcache`). If this is a problem, use:

`ulong hclassno6u_no_cache(ulong d)` as `hclassno6u` without creating caches. Existing caches will be used.

13.3.3 `t_QFB`.

The functions in this section operate on binary quadratic forms of type `t_QFB`. When specified, a `t_QFB` argument q attached to an indefinite form can be replaced by the pair $[q, d]$ where the `t_REAL` d is Shanks's distance.

`GEN qfb_1(GEN q)` given a `t_QFB` q, return the unit form q^0.

`int qfb_equal1(GEN q)` returns 1 if the `t_QFB` q is the unit form.

13.3.3.1 Reduction.

`GEN qfbred(GEN x)` reduction of a `t_QFB` x. Also allow extended indefinite forms.

`GEN qfbred_i(GEN x)` internal version of `qfbred`: assume x is a `t_QFB`.

`GEN qfi_red(GEN x)` internal version of `qfbred`: assume x is an imaginary `t_QFB`.

`GEN qfr_red(GEN x)` internal version of `qfbred`: assume x is a real `t_QFB` and returns a `t_QFB`.

`GEN qfr_cf(GEN x)` internal version of `contfrac(x)`: assume x is a real `t_QFB`.

`GEN qfr_boundcf(GEN x, long n)` internal version of `contfrac(x,,n)`: assume x is a real `t_QFB`.

13.3.3.2 Composition.

`GEN qfbcomp(GEN x, GEN y)` compose the two `t_QFB` x and y (with same discriminant), then reduce the result. This is the same as `gmul(x,y)`. Also allow extended indefinite forms.

`GEN qfbcomp_i(GEN x, GEN y)` internal version of `qfbcomp`: assume x and y are `t_QFB` of the same discriminant.

`GEN qfbsqr(GEN x)` as `qfbcomp(x,x)`.

`GEN qfbsqr_i(GEN x)` as `qfbcomp_i(x,y)`.

Same as above, *without* reducing the result:

`GEN qfbcompraw(GEN x, GEN y)` compose two `t_QFB`s, without reducing the result. Also allow extended indefinite forms.

`GEN qfbcompraw_i(GEN x, GEN y)` internal version of `qfbcompraw`: assume x and y are `t_QFB` of the same discriminant.

13.3.3.3 Powering.

`GEN qfbpow(GEN x, GEN n)` computes x^n and reduce the result. Also allow extended indefinite forms.

`GEN qfbpows(GEN x, long n)` computes x^n and reduce the result. Also allow extended indefinite forms.

`GEN qfbpow_i(GEN x, GEN n)` internal version of `qfbcomp`. Assume x is a `QFB`.

`GEN qfbpowraw(GEN x, long n)` compute x^n (pure composition, no reduction), for a `t_QFB` x. Also allow indefinite forms.

13.3.3.4 Order and discrete log for imaginary qfb.

`GEN qfi_order(GEN q, GEN o)` assuming that the imaginary `t_QFB` q has order dividing o, compute its order in the class group. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

`GEN qfi_log(GEN a, GEN g, GEN o)` given an imaginary `t_QFB` a and assuming that the `t_QFB` g has order o, compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Uses a generic Pollig-Hellman algorithm, then either Shanks (small o) or Pollard rho (large o) method. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

`GEN qfi_Shanks(GEN a, GEN g, long n)` given an imaginary `t_QFB` a and assuming that the `t_QFB` g has (small) order n, compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Directly uses Shanks algorithm, which is inefficient when n is composite.

13.3.3.5 Solve, Cornacchia.

The following functions underly `qfbsolve`; p denotes a prime number.

`GEN qfisolvep(GEN Q, GEN p)` solves $Q(x,y) = p$ over the integers, for an imaginary `t_QFB` Q. Return `gen_0` if there are no solutions.

`GEN qfrsolvep(GEN Q, GEN p)` solves $Q(x,y) = p$ over the integers, for a real `t_QFB` Q. Return `gen_0` if there are no solutions.

`long cornacchia(GEN d, GEN p, GEN *px, GEN *py)` solves $x^2 + dy^2 = p$ over the integers, where $d > 0$ is congruent to 0 or 3 modulo 4. Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

`long cornacchia2(GEN d, GEN p, GEN *px, GEN *py)` as `cornacchia`, for the equation $x^2+dy^2 = 4p$.

`long cornacchia2_sqrt(GEN d, GEN p, GEN b, GEN *px, GEN *py)` as `cornacchia2`, where $p > 2$ and b is the smallest squareroot of d modulo p.

13.3.3.6 Prime forms.

`GEN primeform_u(GEN D, ulong p)` `t_QFB` of discriminant D whose first coefficient is the prime p, assuming $(D/p) \geq 0$.

`GEN primeform(GEN D, GEN p)`

13.3.4 Efficient real quadratic forms. Unfortunately, real `t_QFB`s are very inefficient, and are only provided for backward compatibility.

• they do not contain needed quantities, which are thus constantly recomputed (the discriminant square root $\sqrt{D}$ and its integer part),

• the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

• `qfr3`, a container $[a,b,c]$ with at least 3 entries: the three coefficients; the idea is to ignore the distance component.

• `qfr5`, a container with at least 5 entries $[a,b,c,e,d]$: the three coefficients a `t_REAL` d and a `t_INT` e coding the distance component $2^{Ne}d$, in exponential form, for some large fixed N.

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFB` will do whenever a `qfr3` is expected. Routines using these objects require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
  GEN D;        /* discriminant, t_INT   */
  GEN sqrtD;    /* sqrt(D), t_REAL       */
  GEN isqrtD;   /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant $D > 0$, initialize S for computations at precision `prec` ($\sqrt{D}$ is computed to that initial accuracy).

All functions below are shallow, and not stack-clean.

`GEN qfr3_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr3`, reducing the result.

`GEN qfr3_compraw(GEN x, GEN y)` as `qfr3_comp`, without reducing the result.

`GEN qfr3_pow(GEN x, GEN n, struct qfr_data *S)` compute x^n, reducing along the way.

`GEN qfr3_red(GEN x, struct qfr_data *S)` reduce x.

`GEN qfr3_rho(GEN x, struct qfr_data *S)` perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

`GEN qfr3_to_qfr(GEN x, GEN d)` recover an ordinary `t_QFB` from the `qfr3` x, adding disriminant component d.

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form (a, b, c) to $\rho(a, b, c) = (c, -b \bmod 2c, *)$; the distance component is multiplied by (a floating point approximation to) $(b + \sqrt{D})/(b - \sqrt{D})$.

`GEN qfr5_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr5`, reducing the result, and updating the distance component.

`GEN qfr5_compraw(GEN x, GEN y)` as `qfr5_comp`, without reducing the result.

`GEN qfr5_pow(GEN x, GEN n, struct qfr_data *S)` compute x^n, reducing along the way.

`GEN qfr5_red(GEN x, struct qfr_data *S)` reduce x.

`GEN qfr5_rho(GEN x, struct qfr_data *S)` perform one reduction step.

`GEN qfr5_dist(GEN e, GEN d, long prec)` decode the distance component from exponential (`qfr5`-specific) to logarithmic form (true Shanks's distance).

`GEN qfr_to_qfr5(GEN x, long prec)` convert a real `t_QFB` to a `qfr5` with initial trivial distance component ($= 1$).

`GEN qfr5_to_qfr(GEN x, GEN d)`, assume x is a `qfr5` and d is `NULL` or the original distance component of some real `t_QFB`. Convert x to a `t_QFB`, with the correct (logarithmic) distance component if d is not `NULL`.

13.4 Linear algebra over Z.

13.4.1 Hermite and Smith Normal Forms.

`GEN ZM_hnf(GEN x)` returns the upper triangular Hermite Normal Form of the `ZM` x (removing 0 columns), using the `ZM_hnfall` algorithm. If you want the true HNF, use `ZM_hnfall(x, NULL, 0)`.

`GEN ZM_hnfmod(GEN x, GEN d)` returns the HNF of the `ZM` x (removing 0 columns), assuming the `t_INT` d is a multiple of the determinant of x. This is usually faster than `ZM_hnf` (and uses less memory) if the dimension is large, > 50 say.

`GEN ZM_hnfintersectmod(GEN L1, GEN L2, GEN d)` returns the HNF of the intersection of the $\mathbf{Z}$-modules with bases `L1` and `L2`, assuming that this intersection contains $d\mathbf{Z}^n$.

`GEN ZM_hnfmodid(GEN x, GEN d)` returns the HNF of the `ZM` x concatenated with the diagonal matrix with diagonal d, where d is a vector of integers of compatible dimension. Variant: if d is a `t_INT`, then concatenate dId.

`GEN ZM_hnfmodprime(GEN x, GEN p)` returns the HNF of the matrix $(x \mid p\mathrm{Id})$ (removing 0 columns), for a `ZM` x and a prime number p. The algorithm involves only $\mathbf{F}_p$-linear algebra and is is faster than `ZM_hnfmodid` (which will call it when d is prime).

`GEN ZM_hnfmodall(GEN x, GEN d, long flag)` low-level function underlying the `ZM_hnfmod` variants. If `flag` is 0, calls `ZM_hnfmod(x,d)`; `flag` is an or-ed combination of:

• `hnf_MODID` call `ZM_hnfmodid` instead of `ZM_hnfmod`,

• `hnf_PART` return as soon as we obtain an upper triangular matrix, saving time. The pivots are nonnegative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.

• `hnf_CENTER` returns the centered HNF, where the entries to the right of a pivot p are centered residues in $[-p/2, p/2[$, hence smallest possible in absolute value, but possibly negative.

`GEN ZM_hnfmodall_i(GEN x, GEN d, long flag)` as `ZM_hnfmodall` without final garbage collection. Not stack-clean.

`GEN ZM_hnfall(GEN x, GEN *U, long remove)` returns the upper triangular HNF H of the `ZM` x; if U is not `NULL`, set if to the matrix U such that $xU = H$. If `remove` $= 0$, H is the true HNF, including 0 columns; if `remove` $= 1$, delete the 0 columns from H but do not update U accordingly (so that the integer kernel may still be recovered): we no longer have $xU = H$; if `remove` $= 2$, remove 0 columns from H and update U so that $xU = H$. The matrix U is square and invertible unless `remove` $= 2$.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix U may be very large, when the kernel is large.

`GEN ZM_hnfall_i(GEN x, GEN *U, long remove)` as `ZM_hnfall` without final garbage collection. Not stack-clean.

`GEN ZM_hnfperm(GEN A, GEN *ptU, GEN *ptperm)` returns the hnf $H = PAU$ of the matrix PA, where P is a suitable permutation matrix, and $U \in \mathrm{Gl}_n(\mathbf{Z})$. P is chosen so as to (heuristically) minimize the size of U; in this respect it is less efficient than `ZM_hnflll` but usually faster. Set `*ptU` to U and `*pterm` to a `t_VECSMALL` representing the row permutation attached to $P = (\delta_{i,\mathtt{perm}[i]}$. If

`ptU` is set to `NULL`, U is not computed, saving some time; although useless, setting `ptperm` to `NULL` is also allowed.

`GEN ZM_hnf_knapsack(GEN x)` given a `ZM` x, compute its HNF h. Return h if it has the knapsack property: every column contains only zeroes and ones and each row contains a single 1; return `NULL` otherwise. Not stack-clean.

`GEN ZM_hnflll(GEN x, GEN *U, int remove)` returns the HNF H of the `ZM` x; if U is not `NULL`, set if to the matrix U such that $xU = H$. The meaning of `remove` is the same as in `ZM_hnfall`.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix U is essentially optimally small with respect to the L_2 norm.

`GEN ZM_hnfcenter(GEN M)`. Given a `ZM` in HNF M, update it in place so that nondiagonal entries belong to a system of *centered* residues. Not stack-clean.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

`GEN hnf_divscale(GEN A, GEN B, GEN t)` A an upper triangular `ZM`, B a `ZM`, t an integer, such that $C := tA^{-1}B$ is integral. Return C.

`GEN hnf_invscale(GEN A, GEN t)` A an upper triangular `ZM`, t an integer such that $C := tA^{-1}$ is integral. Return C. Special case of `hnf_divscale` when B is the identity matrix.

`GEN hnf_solve(GEN A, GEN B)` A a `ZM` in upper HNF (not necessarily square), B a `ZM` or `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`GEN hnf_invimage(GEN A, GEN b)` A a `ZM` in upper HNF (not necessarily square), b a `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`int hnfdivide(GEN A, GEN B)` A and B are two upper triangular `ZM`. Return 1 if $A^{-1}B$ is integral, and 0 otherwise.

`GEN ZC_modhnf(GEN A, GEN H, GEN *pQ)` reduce the integral column vector A modulo the HNF matrix H (of compatible dimensions). The reduction R is a `ZC` satisfying $0 \leq R_i < H_{i,i}$ for all i. If `pQ` is not `NULL`, `*pQ` is set to the `ZC` Q such that $A = HQ + R$. Both Q and R are unique.

The function does not in fact assume that H is in HNF, it needs only be upper triangular with positive diagonal entries.

`GEN ZM_modhnf(GEN A, GEN H, GEN *Q)` reduces the integral matrix A modulo the HNF matrix H (of compatible dimensions). This is the same as applying `ZC_modhnf` to successive columns of A. We still have $A = HQ + R$ as a matrix equality, with unique Q and R, both of them `ZM`.

Smith Normal Form.

`GEN ZM_snf(GEN x)` returns the Smith Normal Form (vector of elementary divisors) of the `ZM` x.

`GEN ZM_snfall(GEN x, GEN *U, GEN *V)` returns `ZM_snf(x)` and sets U and V to unimodular matrices such that $U\,x\,V = D$ (diagonal matrix of elementary divisors). Either (or both) U or V may be `NULL` in which case the corresponding matrix is not computed.

`GEN ZV_snfall(GEN d, GEN *U, GEN *V)` here d is a `ZV`; same as `ZM_snfall` applied to `diagonal(d)`, but faster.

`GEN ZM_snfall_i(GEN x, GEN *U, GEN *V, long flag)` low level version of `ZM_snfall`:

• if the first bit of *flag* is 0, return a diagonal matrix (as in `ZM_snfall`), else a vector of elementary divisors (as in `ZM_snf`).

• if the second bit of *flag* is 1, assume that x is invertible and allow U and V to have determinant congruent to 1 modulo d, where d is the largest elementary divisor of x. Rationale: the finite group $G = \mathbf{Z}^n/\Im x$ has exponent d and we are only interested in the action of U, V as they act on G not in genuine unimodular matries. (See `ZM_snf_group`.)

`void ZM_snfclean(GEN d, GEN U, GEN V)` assuming d, U, V come from `d = ZM_snfall(x, &U, &V)`, where U or V may be `NULL`, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and U, V are updated. This also works when d is a `t_VEC` of elementary divisors. Shallow function.

`void ZV_snfclean(GEN d)` assuming d is a `t_VEC` of elementary divisors, return a shortened version where divisors equal to 1 are deleted. Shallow function; we return d itself if no divisor is 1.

`void ZV_snf_trunc(GEN D)` given a vector D of elementary divisors (i.e. a `ZV` such that $d_i \mid d_{i+1}$), truncate it *in place* to leave out the trivial divisors (equal to 1).

`GEN ZM_snf_group(GEN H, GEN *U, GEN *Uinv)` this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators $g = (g_1, \dots, g_n)$ and a vector $X = (x_i) \in \mathbf{Z}^n$, we write gX for the group element $\sum_i x_i g_i$; analogously if M is an $n \times r$ integer matrix gM is a vector containing r group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix H among the generators, i.e. a `ZM` (not necessarily square) such that $gX = 0$ for some $X \in \mathbf{Z}^n$ if and only if X is in the integer image of H, so that the abelian group is isomorphic to $\mathbf{Z}^n/\mathrm{Im}H$. *The routine assumes that H is in HNF;* replace it by its HNF if it is not the case. (Of course this defines the same group.)

Let G a minimal system of generators in SNF for our abstract group: if the d_i are the elementary divisors $(\dots \mid d_2 \mid d_1)$, each G_i has either infinite order ($d_i = 0$) or order $d_i > 1$. Let D the matrix with diagonal (d_i), then

$$GD = 0, \quad G = gU_{\mathrm{inv}}, \quad g = GU,$$

for some integer matrices U and U_{inv}. Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy $D \mid (UU_{\mathrm{inv}} - \mathrm{Id})$ and $H \mid (U_{\mathrm{inv}}U - \mathrm{Id})$.

The function returns the vector of elementary divisors (d_i); if `U` is not `NULL`, it is set to U; if `Uinv` is not `NULL` it is set to U_{inv}. The function is not stack-clean.

`GEN ZV_snf_group(GEN d, GEN *newU, GEN *newUi)`, here d is a `ZV`; same as `ZM_snf_group` applied to `diagonal(d)`, but faster.

`GEN ZV_snf_gcd(GEN v, GEN N)` given a vector v of integers and a positive integer N, return the vector whose entries are the gcds $(v[i], N)$. Use case: if v gives the cyclic components for some abelian group G of finite type, then this returns the structure of the finite groupe G/G^N.

The following functions compute the p^n-rank of abelian groups given a vector of elementary divisors and underly `snfrank`:

`long ZV_snf_rank(GEN D, GEN p)` assume D is a `ZV` and p is a `t_INT`.

`long ZV_snf_rank_u(GEN D, ulong p)` assume D is a `ZV`.

`long zv_snf_rank(GEN D, ulong p)` assume D is a `zv`.

The following routines underly the various `matrixqz` variants. In all case the $m \times n$ `t_MAT` x is assumed to have rational (`t_INT` and `t_FRAC`) coefficients

`GEN QM_ImQ(GEN x)` returns a basis for $\mathrm{Im}_{\mathbf{Q}} x \cap \mathbf{Z}^n$.

`GEN QM_ImZ(GEN x)` returns a basis for $\mathrm{Im}_{\mathbf{Z}} x \cap \mathbf{Z}^n$.

`GEN QM_ImQ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Q}} x \cap \mathbf{Z}^n$.

`GEN QM_ImZ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Z}} x \cap \mathbf{Z}^n$.

`GEN QM_ImQ_hnfall(GEN A, GEN *pB, long remove)` as `QM_ImQ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

`GEN QM_ImZ_hnfall(GEN A, GEN *pB, long remove)` as `QM_ImZ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

`GEN QM_ImQ_all(GEN A, GEN *pB, long remove, long hnf)` as `QM_ImQ`, further returning the transformation matrix as in `ZM_hnfall`, and returning an HNF basis if `hnf` is nonzero.

`GEN QM_ImZ_all(GEN A, GEN *pB, long remove, long hnf)` as `QM_ImZ`, further returning the transformation matrix as in `ZM_hnfall`, and returning an HNF basis if `hnf` is nonzero.

`GEN QM_minors_coprime(GEN x, GEN D)`, assumes $m \geq n$, and returns a matrix in $M_{m,n}(\mathbf{Z})$ with the same $\mathbf{Q}$-image as x, such that the GCD of all $n \times n$ minors is coprime to D; if D is `NULL`, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

`GEN hnf(GEN x)` checks whether x is a `ZM`, then calls `ZM_hnf`. Normally useless in library mode.

`GEN hnfmod(GEN x, GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmod`. Normally useless in library mode.

`GEN hnfmodid(GEN x, GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmodid`. Normally useless in library mode.

`GEN hnfall(GEN x)` calls `ZM_hnfall(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnflll(GEN x)` calls `ZM_hnflll(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnfperm(GEN x)` calls `ZM_hnfperm(x, &U, &P)` and returns $[H, U, P]$. Normally useless in library mode.

`GEN smith(GEN x)` checks whether x is a `ZM`, then calls `ZM_snf`. Normally useless in library mode.

`GEN smithall(GEN x)` checks whether x is a `ZM`, then calls `ZM_snfall(x, &U, &V)` and returns $[U, V, D]$. Normally useless in library mode.

Some related functions over $K[X]$, K a field:

`GEN gsmith(GEN A)` the input matrix must be square, returns the elementary divisors.

`GEN gsmithall(GEN A)` the input matrix must be square, returns the $[U, V, D]$, D diagonal, such that $UAV = D$.

`GEN RgM_hnfall(GEN A, GEN *pB, long remove)` analogous to `ZM_hnfall`.

`GEN smithclean(GEN z)` cleanup the output of `smithall` or `gsmithall` (delete elementary divisors equal to 1, updating base change matrices).

13.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

- `GEN qflll0(GEN x, long flag)`

`GEN lll(GEN x)` *flag*$= 0$

`GEN lllint(GEN x)` *flag*$= 1$

`GEN lllkerim(GEN x)` *flag*$= 4$

`GEN lllkerimgen(GEN x)` *flag*$= 5$

`GEN lllgen(GEN x)` *flag*$= 8$

- `GEN qflllgram0(GEN x, long flag)`

`GEN lllgram(GEN x)` *flag*$= 0$

`GEN lllgramint(GEN x)` *flag*$= 1$

`GEN lllgramkerim(GEN x)` *flag*$= 4$

`GEN lllgramkerimgen(GEN x)` *flag*$= 5$

`GEN lllgramgen(GEN x)` *flag*$= 8$

The basic workhorse underlying all integral and floating point LLLs is

`GEN ZM_lll(GEN x, double D, long flag)`, where x is a `ZM`; $D \in]1/4, 1[$ is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value: $D = 0.99$).

Important. This function is not stack-clean. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

• `LLL_GRAM`. If set, the input matrix x is the Gram matrix ${}^t vv$ of some lattice vectors v.

• `LLL_INPLACE`. Incompatible with `LLL_GRAM`. If unset, we return the base change matrix U, otherwise the transformed matrix xU. Implies `LLL_IM` (see below).

• `LLL_KEEP_FIRST`. The first vector b_1 in the output basis is the same one as was originally input. In particular, if b_1 cannot be completed into a basis, the behavior is undefined. Provided b_1 is a shortest nonzero vector of the lattice, the output basis is still LLL-reduced. This flag is used to reduce maximal orders of number fields with respect to the T_2 quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).

• `LLL_COMPATIBLE`. DEPRECATED. This is now a no-op.

The last three flags are mutually exclusive, either 0 or a single one must be set:

• `LLL_KER` If set, only return a kernel basis K (not LLL-reduced).

• `LLL_IM` If set, only return an LLL-reduced lattice basis T. (This is implied by `LLL_INPLACE`).

• `LLL_ALL` If set, returns a 2-component vector $[K, T]$ corresponding to both kernel and image.

`GEN lllfp(GEN x, double D, long flag)` is a variant for matrices with inexact entries: x is a matrix with real coefficients (types `t_INT`, `t_FRAC` and `t_REAL`), D and *flag* are as in `ZM_lll`. The matrix is rescaled, rounded to nearest integers, then fed to `ZM_lll`. The flag `LLL_INPLACE` is still accepted but less useful (it returns an LLL-reduced basis attached to rounded input, instead of an exact base change matrix).

`GEN ZM_lll_norms(GEN x, double D, long flag, GEN *ptB)` slightly more general version of `ZM_lll`, setting `*ptB` to a vector containing the squared norms of the Gram-Schmidt vectors (b_i^*) attached to the output basis (b_i), $b_i^* = b_i + \sum_{j<i} \mu_{i,j} b_j^*$.

`GEN lllintpartial_inplace(GEN x)` given a `ZM` x of maximal rank, returns a partially reduced basis (b_i) for the space spanned by the columns of x: $|b_i \pm b_j| \geq |b_i|$ for any two distinct basis vectors b_i, b_j. This is faster than the LLL algorithm, but produces much larger bases.

`GEN lllintpartial(GEN x)` as `lllintpartial_inplace`, but returns the base change matrix U from the canonical basis to the b_i, i.e. xU is the output of `lllintpartial_inplace`.

`GEN RM_round_maxrank(GEN G)` given a matrix G with real floating point entries and independent columns, let G_e be the rescaled matrix $2^e G$ rounded to nearest integers, for $e \geq 0$. Finds a small e such that the rank of G_e is equal to the rank of G (its number of columns) and return G_e. This is useful as a preconditioning step to speed up LLL reductions, see `nf_get_Gtwist`. Suitable for `gc_upto`, but does not collect garbage.

`GEN Hermite_bound(long n, long prec)` return a majoration of γ_n^n where γ_n is the Hermite constant for lattices of dimension n. The bound is sharp in dimension $n \leq 8$ and $n = 24$.

13.4.3 Linear dependencies.

The following functions underly the `lindep` GP function:

`GEN lindep(GEN v)` real/complex entries, guess that about only the 80% leading bits of the input are correct.

`GEN lindep_bit(GEN v, long b)` real/complex entries, explicit form of the above: multiply the input by 2^b and round to nearest integer before looking for a linear dependency. Truncating dubious bits allows to find better relations.

`GEN lindepfull_bit(GEN v, long b)` as `lindep_bit` but return a matrix M with $n = \#v$ columns and r rows, with $r = n + 1$ (if v is real) or $n + 2$ (general case) which is an LLL-reduced basis of the lattice formed by concatenating vertically an identity matrix and the floor of 2^b`real`(v) and 2^b`imag`(v) if $r = n+2$. The first n rows of M potentially correspond to relations: whenever the last $r - n$ entries of a column are small. The function `lindep_bit` essentially returns the first column of M truncated to n components.

`GEN lindep_padic(GEN v)` p-adic entries.

`GEN lindep_Xadic(GEN v)` polynomial entries.

`GEN deplin(GEN v)` returns a nonzero kernel vector for a `t_MAT` input.

Deprecated routine:

`GEN lindep2(GEN x, long dig)` analogous to `lindep_bit`, with `dig` counting decimal digits.

13.4.4 Reduction modulo matrices.

`GEN ZC_hnfdivrem(GEN x, GEN y, GEN *Q)` assuming y is an invertible `ZM` in HNF and x is a `ZC`, returns the `ZC` R equal to x mod y (whose i-th entry belongs to $[-y_{i,i}/2, y_{i,i}/2[$). Stack-clean *unless* x is already reduced (in which case, returns x itself, not a copy). If Q is not `NULL`, set it to the `ZC` such that $x = yQ + R$.

`GEN ZM_hnfdivrem(GEN x, GEN y, GEN *Q)` reduce each column of the `ZM` x using `ZC_hnfremdiv`. If Q is not `NULL`, set it to the `ZM` such that $x = yQ + R$.

`GEN ZC_hnfrem(GEN x, GEN y)` alias for `ZC_hnfremdiv(x,y,NULL)`.

`GEN ZM_hnfrem(GEN x, GEN y)` alias for `ZM_hnfremdiv(x,y,NULL)`.

`GEN ZC_reducemodmatrix(GEN v, GEN y)` Let y be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC v modulo the $\mathbf{Z}$-module Y spanned by y : if the columns of y are denoted by $(y_1, \dots, y_{n-1})$, we return $y_n \equiv v$ modulo Y, such that the Gram-Schmidt coefficients $\mu_{n,j}$ are less than $1/2$ in absolute value for all $j < n$. In short, y_n is almost orthogonal to Y.

`GEN ZM_reducemodmatrix(GEN v, GEN y)` Let y be as in `ZC_reducemodmatrix`, and v be a ZM. This returns a matrix v which is congruent to v modulo the $\mathbf{Z}$-module spanned by y, whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

`GEN ZC_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

`GEN ZM_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

`GEN reducemodinvertible(GEN x, GEN y)` y is an invertible matrix and x a `t_COL` or `t_MAT` of compatible dimension. Returns $x - y\lfloor y^{-1}x \rceil$, which has small entries and differs from x by an integral linear combination of the columns of y. Suitable for `gc_upto`, but does not collect garbage.

`GEN closemodinvertible(GEN x, GEN y)` returns $x-$`reducemodinvertible`(x, y), i.e. an integral linear combination of the columns of y, which is close to x.

`GEN reducemodlll(GEN x, GEN y)` LLL-reduce the nonsingular `ZM` y and call `reducemodinvertible` to find a small representative of x mod $y\mathbf{Z}^n$. Suitable for `gc_upto`, but does not collect garbage.

13.5 Finite abelian groups and characters.

13.5.1 Abstract groups.

A finite abelian group G in GP format is given by its Smith Normal Form as a pair $[h, d]$ or triple $[h, d, g]$. Here h is the cardinality of G, (d_i) is the vector of elementary divisors, and (g_i) is a vector of generators. In short, $G = \oplus_{i\le n}(\mathbf{Z}/d_i\mathbf{Z})g_i$, with $d_n \mid \ldots \mid d_2 \mid d_1$ and $\prod d_i = h$.

Let $e(x) := \exp(2i\pi x)$. For ease of exposition, we restrict to complex-valued characters, but everything applies to more general fields K where e denotes a morphism $(\mathbf{Q}, +) \to (K^*, \times)$ such that $e(a/b)$ denotes a b-th root of unity.

A *character* on the abelian group $\oplus(\mathbf{Z}/d_j\mathbf{Z})g_j$ is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j/d_j)$.

A *quasi-normalized character* is of the form $[d, (n_j)]$, such that $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$, where $\zeta_d = e(1/d)$.

A *normalized character* is a quasi-normalized $[d, (n_j)]$, such that d is *minimal*, i.e., d is the order of `chi`.

`GEN cyc_normalize(GEN d)` shallow function. Given a vector $(d_i)_{i\le n}$ of elementary divisors for a finite group (no d_i vanish), returns the vector $D = [1]$ if $n = 0$ (trivial group) and $[d_1, d_1/d_2, \ldots, d_1/d_n]$ otherwise. This will allow to define characters as $\chi(\prod g_j^{x_j}) = e(\sum_j x_j a_j D_j/D_1)$, with a common denominator D_1 see `char_normalize`.

`GEN char_normalize(GEN chi, GEN ncyc)` shallow function. Given a character `chi` $= (a_j)$ and *ncyc* from `cyc_normalize` above, returns the normalized representation $[d, (n_j)]$, such that $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$, where $\zeta_d = e(1/d)$ and d is *minimal*. Shallow function.

`GEN char_simplify(GEN D, GEN N)` given a quasi-normalized character $[D, (N_j)]$ such that $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$, return a normalized character $[d, (n_j)]$ with d *minimal*. Shallow function.

`GEN char_renormalize(GEN nchi, GEN o)` shallow function. Given a quasi-normalized character `nchi` $= [d, (n_j)]$ and a multiple D of d, returns the quasi-normalized representation $[D, (N_j)]$, such that $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$. Shallow function.

`GEN char_denormalize(GEN cyc, GEN d, GEN n)` given a normalized representation $[d, n]$ (where d need not be minimal) of a character on the abelian group with abelian divisors `cyc`, return the

attached character (where the image of each generator g_i is given in terms of roots of unity of different orders `cyc`$[i]$).

`GEN charconj(GEN cyc, GEN chi)` return the complex conjugate of `chi`.

`GEN charmul(GEN cyc, GEN a, GEN b)` return the product character $a \times b$.

`GEN chardiv(GEN cyc, GEN a, GEN b)` returns the character $a/b = a \times \overline{b}$.

`int char_check(GEN cyc, GEN chi)` return 1 if `chi` is a character compatible with cyclic factors `cyc`, and 0 otherwise.

`GEN cyc2elts(GEN d)` given a `t_VEC` $d = (d_1, \dots, d_n)$ of nonnegative integers, return the vector of all `t_VECSMALL`s of length n whose i-th entry lies in $[0, d_i[$. Assumes that the product of the d_i fits in a `long`.

`long zv_cyc_minimize(GEN d, GEN c, GEN coprime)` given $d = (d_1, \dots, d_n)$, $d_n \mid \dots \mid d_1 \neq 0$ a list of elementary divisors for a finite abelian group as a `t_VECSMALL`, given $c = [g_1, \dots, g_n]$ representing an element in the group, and given a mask `coprime` (as from `coprimes_zv`(o)) representing a list of forbidden congruence classes modulo o, return an integer k such that `coprime`$[k\%o]$ is nonzero and $k \cdot c$ is lexicographically minimal. For instance, if c is attached to a Dirichlet character χ of order o via the usual identification $\chi(g_i) = \zeta_{g_i}^{c_i}$, then χ^k is a "canonical" representative in the Galois orbit of χ.

`long zv_cyc_minimal(GEN d, GEN c, GEN coprime)` return 1 if `zv_cyc_minimize` would return $k = 1$, i.e. c is already the canonical representative for the attached character orbit.

13.5.2 Dirichlet characters.

The functions in this section are specific to characters on $(\mathbf{Z}/N\mathbf{Z})^*$. The argument G is a special `bid` structure as returned by `znstar0(N, nf_INIT)`. In this case, there are additional ways to input character via Conrey's representation. The character `chi` is either a `t_INT` (Conrey label), a `t_COL` (a Conrey logarithm) or a `t_VEC` (generic character on `bid.gen` as explained in the previous subsection). The following low-level functions are called by GP's generic character functions.

`int zncharcheck(GEN G, GEN chi)` return 1 if `chi` is a valid character and 0 otherwise.

`GEN zncharconj(GEN G, GEN chi)` as `charconj`.

`GEN znchardiv(GEN G, GEN a, GEN b)` as `chardiv`.

`GEN zncharker(GEN G, GEN chi)` as `charker`.

`GEN znchareval(GEN G, GEN chi, GEN n, GEN z)` as `chareval`.

`GEN zncharmul(GEN G, GEN a, GEN b)` as `charmul`.

`GEN zncharpow(GEN G, GEN a, GEN n)` as `charpow`.

`GEN zncharorder(GEN G, GEN chi)` as `charorder`.

The following functions handle characters in Conrey notation (attached to Conrey generators, not `G.gen`):

`int znconrey_check(GEN cyc, GEN chi)` return 1 if `chi` is a valid Conrey logarithm and 0 otherwise.

`GEN znconrey_normalized(GEN G, GEN chi)` return normalized character attached to `chi`, as in `char_normalize` but on Conrey generators.

`GEN znconreyfromchar(GEN G, GEN chi)` return Conrey logarithm attached to the generic (`t_VEC`, on `G.gen`)

`GEN znconreyfromchar_normalized(GEN G, GEN chi)` return normalized Conrey character attached to the generic (`t_VEC`, on `G.gen`) character `chi`.

`GEN znconreylog_normalize(GEN G, GEN m)` given a Conrey logarithm m (`t_COL`), return the attached normalized Conrey character, as in `char_normalize` but on Conrey generators.

`GEN znchar_quad(GEN G, GEN D)` given a nonzero `t_INT` D congruent to $0, 1 \bmod 4$, return $(D/.)$ as a character modulo N, given by a Conrey logarithm (`t_COL`). Assume that $|D|$ divides N.

`GEN Zideallog(GEN G, GEN x)` return the `znconreylog` of x expressed on `G.gen`, i.e. the ordinary discrete logarithm from `ideallog`.

`GEN ncharvecexpo(GEN G, GEN nchi)` given `nchi` $= [d, n]$ a quasi-normalized character (d may be a multiple of the character order), i.e. $\chi(g_i) = e(n[i]/d)$ for all Conrey or SNF generators g_i (as usual, we use SNF generators if n is a `t_VEC` and the Conrey generators otherwise). Return a `t_VECSMALL` v such that $v[i] = -1$ if $(i, N) > 1$ else $\chi(i) = e(v[i]/d)$, $1 \le i \le N$.

13.6 Hecke characters.

The functions in this section are specific to Hecke characters. The argument `gc` is a `gchar` structure as returned by `gcharinit(bnf, mod)`, and the character `chi` is a `t_COL` of components on the SNF generators of `gc`.

`GEN eulerf_gchar(GEN an, GEN p, long prec)` `an` being the first component of a Hecke L-function `Ldata` (as output by `lfungchar`) and p a prime number, return the Euler factor at p.

`GEN gchari_lfun(GEN gc, GEN chi, GEN w)` `chi` being a `t_VEC` describing a Hecke character encoded on the internal basis `gc[1]`, return the `Ldata` structure corresponding to the Hecke L-function associated to `chi`.

`int is_gchar_group(GEN gc)` return 1 if `gc` is a valid gchar structure and 0 otherwise.

`GEN lfungchar(GEN gc, GEN chi)` return the `Ldata` structure corresponding to the Hecke L-function associated to `chi`.

`GEN vecan_gchar(GEN an, long n, long prec)` `an` being the first component of a Hecke L-function `Ldata` (as output by `lfungchar`), return a `t_VEC` of length n containing the first n Dirichlet coefficients of this L-function, computed to absolute precision `prec`.

13.7 Central simple algebras.

13.7.1 Initialization.

Low-level routines underlying `alginit`; argument `rnf` (resp. `nf`) must be true *rnf* (resp. *nf*) structure. The meaning of *flag* is the same as in `alginit`.

`GEN alg_csa_table(GEN nf, GEN mt, long v, long flag)` central simple algebra defined by a multiplication table over `nf`.

`GEN alg_cyclic(GEN rnf, GEN aut, GEN b, long flag)` cyclic algebra $(L/K, \sigma, b)$.

`GEN alg_hasse(GEN nf, long d, GEN hi, GEN hf, long v, long flag)` algebra defined by local Hasse invariants.

`GEN alg_hilbert(GEN nf, GEN a, GEN b, long v, long flag)` quaternion algebra.

`GEN alg_matrix(GEN nf, long n, long v, long flag)` matrix algebra of degree `n` over `nf`.

`GEN alg_complete(GEN rnf, GEN aut, GEN hf, GEN hi, long flag)` cyclic algebra $(L/K, \sigma, b)$ with b computed from the Hasse invariants.

13.7.2 Type checks.

`void checkalg(GEN a)` raise an exception if a was not initialized by `alginit`.

`void checklat(GEN al, GEN lat)` raise an exception if `lat` is not a valid full lattice in the algebra `al`.

`void checkhasse(GEN nf, GEN hi, GEN hf, long n)` raise an exception if (`hi`, `hf`) do not describe valid Hasse invariants of a central simple algebra of degree `n` over `nf`.

`long alg_type(GEN al)` internal function called by `algtype`: assume `al` was created by `alginit` (thereby saving a call to `checkalg`). Return values are symbolic rather than numeric:

- `al_NULL`: not a valid algebra.
- `al_TABLE`: table algebra output by `algtableinit`.
- `al_CSA`: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- `al_CYCLIC`: central simple algebra output by `alginit` and represented by a cyclic algebra.

`long alg_model(GEN al, GEN x)` given an element x in algebra al, check for inconsistencies (raise a type error) and return the representation model used for x:

- `al_ALGEBRAIC`: `basistoalg` form, algebraic representation.
- `al_BASIS`: `algtobasis` form, column vector on the integral basis.
- `al_MATRIX`: matrix with coefficients in an algebra.
- `al_TRIVIAL`: trivial algebra of degree 1; can be understood as both basis or algebraic form (since $e_1 = 1$).

13.7.3 Shallow accessors.

All these routines assume their argument was initialized by `alginit` and provide minor speedups compared to the GP equivalent. The routines returning a `GEN` are shallow.

`long alg_get_absdim(GEN al)` low-level version of `algabsdim`.

`long alg_get_dim(GEN al)` low-level version of `algdim`.

`long alg_get_degree(GEN al)` low-level version of `algdegree`.

`GEN alg_get_aut(GEN al)` low-level version of `algaut`.

`GEN alg_get_auts(GEN al)`, given a cyclic algebra $al = (L/K, \sigma, b)$ of degree n, returns the vector of σ^i, $1 \leq i < n$.

`GEN alg_get_b(GEN al)` low-level version of `algb`.

`GEN alg_get_basis(GEN al)` low-level version of `algbasis`.

`GEN alg_get_center(GEN al)` low-level version of `algcenter`.

`GEN alg_get_char(GEN al)` low-level version of `algchar`.

`GEN alg_get_hasse_f(GEN al)` low-level version of `alghassef`.

`GEN alg_get_hasse_i(GEN al)` low-level version of `alghassei`.

`GEN alg_get_invbasis(GEN al)` low-level version of `alginvbasis`.

`GEN alg_get_invol(GEN al, GEN x)` low-level version of `alginvol`.

`GEN alg_get_multable(GEN al)` low-level version of `algmultable`.

`GEN alg_get_relmultable(GEN al)` low-level version of `algrelmultable`.

`GEN alg_get_splittingfield(GEN al)` low-level version of `algsplittingfield`.

`GEN alg_get_abssplitting(GEN al)` returns the absolute *nf* structure attached to the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splitpol(GEN al)` returns the relative polynomial defining the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splittingdata(GEN al)` low-level version of `algsplittingdata`.

`GEN alg_get_splittingbasis(GEN al)` the matrix *Lbas* from `algsplittingdata`

`GEN alg_get_splittingbasisinv(GEN al)` the matrix *Lbasinv* from `algsplittingdata`.

`GEN alg_get_tracebasis(GEN al)` returns the traces of the basis elements; used by `algtrace`.

`GEN alglat_get_primbasis(GEN lat)` from the description of `lat` as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns a basis of L.

`GEN alglat_get_scalar(GEN lat)` from the description of `lat` as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns λ.

Accessors for `algmodpr` data (reduction map from algebra to $M_k(\mathbf{F}_{p^m})$, see `algmodprinit`).

`GEN algmodpr_get_pr(GEN data)` corresponding prime ideal.

`long algmodpr_get_k(GEN data)` matrix degree k.

`long algmodpr_get_m(GEN data)` field degree m.

`GEN algmodpr_get_ff(GEN data)` `t_FFELT` 1 in the image finite field.

`GEN algmodpr_get_proj(GEN data)` `t_MAT` representing the projection map.

`GEN algmodpr_get_lift(GEN data)` `t_MAT` representing the lifting map.

`GEN algmodpr_get_tau(GEN data)` multiplication matrix of an anti-uniformizer.

`GEN algmodpr_get_p(GEN data)` prime p.

`GEN algmodpr_get_T(GEN data)` polynomial T defining the image finite field.

13.7.4 Other low-level functions.

`GEN conjclasses_algcenter(GEN cc, GEN p)` low-level function underlying `alggroupcenter`, where `cc` is the output of `groupelts_to_conjclasses`, and p is either `NULL` or a prime number. Not stack-clean.

`GEN algsimpledec_ss(GEN al, long maps)` assuming that `al` is semisimple, returns the second component of `algsimpledec(al,maps)`.

`GEN algnewprec_shallow(GEN al, long prec)` shallow version of `algnewprec`.

Chapter 14:
Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

14.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$. Points in the projective plane are represented as follows: the point at infinity $(0 : 1 : 0)$ is coded as `[0]`, a finite point $(x : y : 1)$ outside the projective line at infinity $z = 0$ is coded as $[x, y]$. Note that other points at infinity than $(0 : 1 : 0)$ cannot be represented; this is harmless, since they do not belong to any of the elliptic curves E above.

Points on the curve are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$ implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

14.1.1 Types of elliptic curves.

We call a 5-tuble as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, attached to the elliptic curve E : either an `ell5`, a pair $[a_4, a_6]$ or a `t_STR` in Cremona's notation, e.g. `"11a1"`. The optional D (`NULL` to omit) describes the domain over which the curve is defined.

14.1.2 Type checking.

`void checkell(GEN e)` raise an error unless e is an *ell*.

`int checkell_i(GEN e)` return 1 if e is an *ell* and 0 otherwise.

`void checkell5(GEN e)` raise an error unless e is an *ell* or an *ell5*.

`void checkellpt(GEN z)` raise an error unless z is a point (either finite or at infinity).

`int checkellpt_i(GEN z)` return 1 if z is a point (either finite or at infinity) and 0 otherwise.

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_NF` a number field;

`t_ELL_Qp` the field of p-adic numbers, for some prime p;

`t_ELL_Fp` a prime finite field, base field elements are represented as `Fp`, i.e. a `t_INT` reduced modulo p;

`t_ELL_Fq` a nonprime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether e is an `ell`, defined over a finite field (either prime or nonprime). Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Q(GEN e)` checks whether e is an `ell`, defined over $\mathbf{Q}$. Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Qp(GEN e)` checks whether e is an `ell`, defined over some $\mathbf{Q}_p$. Otherwise the function raises a `pari_err_TYPE` exception.

`void checkellisog(GEN v)` raise an error unless v is an isogeny, from `ellisogeny`.

14.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

14.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`

14.1.3.2 Curves over Q.

`GEN ellQ_get_N(GEN e)` returns the curve conductor

`void ellQ_get_Nfa(GEN e, GEN *N, GEN *faN)` sets N to the conductor and `faN` to its factorization

`int ell_is_integral(GEN e)` return 1 if e is given by an integral model, and 0 otherwise.

`long ellQ_get_CM(GEN e)` if e has CM by a principal imaginary quadratic order, return its discriminant. Else return 0.

`long ellap_CM_fast(GEN e, ulong p, long CM)` assuming that p does not divide the discriminant of E (in particular, E has good reduction at p), and that `CM` is as given by `ellQ_get_CM`, return the trace of Frobenius for $E/\mathbf{F}_p$. This is meant to quickly compute lots of a_p, esp. when e has CM by a principal quadratic order.

`long ellrootno_global(GEN e)` returns the global root number $c \in \{-1, 1\}$.

`GEN ellheightoo(GEN E, GEN P, long prec)` given $P = [x, y]$ an affine point on E, return

$$\lambda_\infty(P) + \frac{1}{12}\log|\mathrm{disc}E| = \frac{1}{2}\mathrm{real}(z\eta(z)) - \log|\sigma(E, z)| \in \mathbf{R},$$

where $\lambda_\infty(P)$ is the canonical local height at infinity and z is `ellpointtoz`(E, P). This is computed using Mestre's (quadratically convergent) AGM algorithm.

`long ellorder_Q(GEN E, GEN P)` return the order of $P \in E(\mathbf{Q})$, using the impossible value 0 for a point of infinite order. Ultimately called by the generic `ellorder` function.

`GEN point_to_a4a6(GEN E, GEN P, GEN p, GEN *a4)` given $E/\mathbf{Q}$, $p \neq 2, 3$ not dividing the discriminant of E and $P \in E(\mathbf{Q})$ outside the kernel of reduction, return the image of P on the short Weierstrass model $y^2 = x^3 + a_4x + a_6$ isomorphic to the reduction E_p of E at p. Also set `a4` to the a_4 coefficient in the above model. This function allows quick computations modulo varying primes p, avoiding the overhead of `ellinit`(E, p), followed by a change of coordinates. It produces data suitable for `FpE` routines.

`GEN point_to_a4a6_Fl(GEN E, GEN P, ulong p, ulong *pa4)` as `point_to_a4a6`, returning a `Fle`.

`GEN elldatagenerators(GEN E)` returns generators for $E(\mathbf{Q})$ extracted from Cremona's table.

`GEN ellanal_globalred(GEN e, GEN *v)` takes an *ell* over $\mathbf{Q}$ and returns a global minimal model E (in `ellinit` form, over $\mathbf{Q}$) for e suitable for analytic computations related to the curve L series: it contains `ellglobalred` data, as well as global and local root numbers. If `v` is not `NULL`, set `*v` to the needed change of variable: `NULL` if e was already the standard minimal model, such that $E =$ `ellchangecurve`$(\mathtt{e}, \mathtt{v})$ otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve e is updated as well with the same information.

`GEN ellanal_globalred_all(GEN e, GEN *v, GEN *N, GEN *tam)` as `ellanal_globalred`; further set `*N` to the curve conductor and `*tam` to the product of the local Tamagawa numbers, including the factor at infinity (multiply by the number of connected components of $e(\mathbf{R})$).

`GEN ellintegralmodel(GEN e, GEN *pv)` return an integral model for e (in `ellinit` form, over $\mathbf{Q}$). Set $v =$ `NULL` (already integral, we returned e itself), else to the variable change $[u, 0, 0, 0]$ making e integral. We have $u = 1/t$, $t > 1$.

`GEN ellintegralmodel_i(GEN e, GEN *pv)` shallow version of `ellintegralmodel`.

`GEN ellQtwist_bsdperiod(GEN E, long s)` let E be a rational elliptic curve given by a minimal model, Λ_E its period lattice, and $s \in \{-1, 1\}$. Let $\Omega_E^{\pm}$ be the canonical periods in $\sqrt{\pm 1}\mathbf{R}^+$ generating $\Lambda_E \cap \sqrt{\pm 1}\mathbf{R}$. Return Ω_E^+ if $s = 1$ and Ω_E^- if $s = -1$.

`GEN elltors_psylow(GEN e, ulong p)` as `elltors`, but return the p-Sylow subgroup of the torsion group.

`GEN elleulerf(GEN E, GEN p)` returns the Euler factor at p of the L-function associated to the curve E defined over a number field.

Deprecated routines.

`GEN elltors0(GEN e, long flag)` this function is deprecated; use `elltors`

14.1.3.3 Curves over a number field *nf*.

Let K be the number field over which E is defined, given by a *nf* or *bnf* structure.

`GEN ellnf_get_nf(GEN E)` returns the underlying `nf`.

`GEN ellnf_get_bnf(GEN x)` returns `NULL` if K does not contain a *bnf* structure, else return the *bnf*.

`GEN ellnf_vecarea(GEN E)` returns the vector of the period lattices areas of all the complex embeddings of `E` in the same order as `E.nf.roots`.

`GEN ellnf_veceta(GEN E)` returns the vector of the quasi-periods of all the complex embeddings of `E` in the same order as `E.nf.roots`.

`GEN ellnf_vecomega(GEN E)` returns the vector of the periods of all the complex embeddings of `E` in the same order as `E.nf.roots`.

14.1.3.4 Curves over $\mathbf{Q}_p$.

`GEN ellQp_get_p(GEN E)` returns p

`long ellQp_get_prec(GEN E)` returns the default p-adic accuracy to which we must compute approximate results attached to E.

`GEN ellQp_get_zero(GEN x)` returns $O(p^n)$, where n is the default p-adic accuracy as above.

The following functions are only defined when E has multiplicative reduction (Tate curves):

`GEN ellQp_Tate_uniformization(GEN E, long prec)` returns a `t_VEC` containing $u^2, u, q, [a, b]$, at p-adic precision `prec`.

`GEN ellQp_u(GEN E, long prec)` returns u.

`GEN ellQp_u2(GEN E, long prec)` returns u^2.

`GEN ellQp_q(GEN E, long prec)` returns the Tate period q.

`GEN ellQp_ab(GEN E, long prec)` returns $[a, b]$.

`GEN ellQp_AGM(GEN E, long prec)` returns $[a, b, R, v]$, where v is an integer, a, b, R are vectors describing the sequence of 2-isogenous curves $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$, $i \geq 1$ converging to the singular curve $E_\infty : y^2 = x^2(x + M)$. We have $a[i] = A[i]p^v$, $b[i] = B[i]p^v$, $R[i] = A_i - B_i$. These are used in `ellpointtoz` and `ellztopoint`.

`GEN ellQp_L(GEN E, long prec)` returns the $\mathcal{L}$-invariant L.

`GEN ellQp_root(GEN E, long prec)` returns e_1.

14.1.3.5 Curves over a finite field $\mathbf{F}_q$.

`GEN ellff_get_p(GEN E)` returns the characteristic

`GEN ellff_get_field(GEN E)` returns p if $\mathbf{F}_q$ is a prime field, and a `t_FFELT` belonging to $\mathbf{F}_q$ otherwise.

`GEN ellff_get_card(GEN E)` returns $\#E(\mathbf{F}_q)$

`GEN ellff_get_gens(GEN E)` returns a minimal set of generators for $E(\mathbf{F}_q)$.

`GEN ellff_get_group(GEN E)` returns `ellgroup`(E).

`GEN ellff_get_m(GEN E)` returns the `t_INT` m as needed by the `gen_ellgroup` function (the order of the pairing required to verify a generating set).

`GEN ellff_get_o(GEN E)` returns $[d,$`factord`$]$, where d is the exponent of $E(\mathbf{F}_q)$.

`GEN ellff_get_D(GEN E)` returns the elementary divisors for $E(\mathbf{F}_q)$ in a form suitable for `gen_ellgens`: either $[d_1]$ or $[d_1,d_2]$, where d_1 is in `elff_get_o` format.

$[d,$`factord`$]$, where d is the exponent of $E(\mathbf{F}_q)$.

`GEN ellff_get_a4a6(GEN E)` returns a canonical "short model" for E, and the corresponding change of variable $[u,r,s,t]$. For $p\neq 2,3$, this is $[A_4,A_6,[u,r,s,t]]$, corresponding to $y^2=x^3+A_4x+A_6$, where $A_4=-27c_4$, $A_6=-54c_6$, $[u,r,s,t]=[6,3b_2,3a_1,108a_3]$.

• If $p=3$ and the curve is ordinary ($b_2\neq 0$), this is $[[b_2],A_6,[1,v,-a_1,-a_3]]$, corresponding to

$$y^2=x^3+b_2x^2+A_6,$$

where $v=b_4/b_2$, $A_6=b_6-v(b_4+v^2)$.

• If $p=3$ and the curve is supersingular ($b_2=0$), this is $[-b_4,b_6,[1,0,-a_1,-a_3]]$, corresponding to

$$y^2=x^3+2b_4x+b_6.$$

• If $p=2$ and the curve is ordinary ($a_1\neq 0$), return $[A_2,A_6,[a_1^{-1},da_1^{-2},0,(a_4+d^2)a_1^{-1}]]$, corresponding to

$$y^2+xy=x^3+A_2x^2+A_6,$$

where $d=a_3/a_1$, $a_1^2A_2=(a_2+d)$ and

$$a_1^6A_6=d^3+a_2d^2+a_4d+a_6+(a_4^2+d^4)a_1^{-2}.$$

• If $p=2$ and the curve is supersingular ($a_1=0$, $a_3\neq 0$), return $[[a_3,A_4,1/a_3],A_6,[1,a_2,0,0]]$, corresponding to

$$y^2+a_3y=x^3+A_4x+A_6,$$

where $A_4=a_2^2+a_4$, $A_6=a_2a_4+a_6$. The value $1/a_3$ is included in the vector since it is frequently needed in computations.

14.1.3.6 Curves over C. (This includes curves over **Q**!)

`long ellR_get_prec(GEN E)` return the default accuracy to which we must compute approximate results attached to E.

`GEN ellR_ab(GEN E, long prec)` return $[a, b]$

`GEN ellR_omega(GEN x, long prec)` return periods $[\omega_1, \omega_2]$.

`GEN ellR_eta(GEN E, long prec)` return quasi-periods $[\eta_1, \eta_2]$.

`GEN ellR_area(GEN x, long prec)` return the area $(\Im(\omega_1\overline{\omega_2}))$.

`GEN ellR_roots(GEN E, long prec)` return $[e_1, e_2, e_3]$. If E is defined over **R**, then e_1 is real. If furthermore disc$E > 0$, then $e_1 > e_2 > e_3$.

`long ellR_get_sign(GEN E)` if E is defined over **R** returns the signe of its discriminant, otherwise return 0.

14.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point z is the point at infinity.

`GEN ellinf()` returns the point at infinity `[0]`.

14.1.5 Change of variables.

`GEN ellchangeinvert(GEN w)` given a change of variables $w = [u, r, s, t]$, returns the inverse change of variables w', such that if $E' =$ `ellchangecurve`$(\mathtt{E}, \mathtt{w})$, then $E =$ `ellchangecurve`(E, w').

14.1.6 Generic helper functions.

The naming scheme assumes an affine equation $F(x, y) = f(x) - (y^2 + h(x)y) = 0$ in standard Weierstrass form: $f = x^3 + a_2x^2 + a_4x + a_6$, $h = a_1x + a_3$. Unless mentionned otherwise, these routine assume that all arguments are compatible with generic functions of `gadd` or `gmul` type. In particular they do not handle elements in number field in `nfalgtobasis` format.

`GEN ellbasechar(GEN E)` returns the characteristic of the base ring over which E is defined.

`GEN ec_bmodel(GEN E, long v)` returns the polynomial $4x^3 + b_2x^2 + 2b_4x + b_6$ in the variable `v`.

`GEN ec_phi2(GEN E, long v)` returns the polynomial $x^4 - b_4x^2 - 2b_6X - b_8$ in the variable `v`.

`GEN ec_f_evalx(GEN E, GEN x)` returns $f(x)$.

`GEN ec_h_evalx(GEN E, GEN x)` returns $h(x)$.

`GEN ec_dFdx_evalQ(GEN E, GEN Q)` returns $3x^2 + 2a_2x + a_4 - a_1y$, where $Q = [x, y]$.

`GEN ec_dFdy_evalQ(GEN E, GEN Q)` returns $-(2y + a_1x + a_3)$, where $Q = [x, y]$.

`GEN ec_dmFdy_evalQ(GEN e, GEN Q)` returns $2y + a_1x + a_3$, where $Q = [x, y]$.

`GEN ec_2divpol_evalx(GEN E, GEN x)` returns $4x^3 + b_2x^2 + 2b_4x + b_6$. This function supports inputs in `nfalgtobasis` format.

`GEN ec_half_deriv_2divpol(GEN E, long v)` returns $6x^2 + b_2x + b_4$ as a `t_POL` in the variable `v`.

`GEN ec_half_deriv_2divpol_evalx(GEN E, GEN x)` returns $6x^2 + b_2x + b_4$.

`GEN ec_3divpol_evalx(GEN E, GEN x)` returns $3x^4 + b_2x^2 + 3b_4x^2 + 3b_6x + b_8$.

14.1.7 Functions to handle elliptic curves over finite fields.

14.1.7.1 Tolerant routines.

`GEN ellap(GEN E, GEN p)` given a prime number p and an elliptic curve defined over $\mathbf{Q}$ or $\mathbf{Q}_p$ (assumed integral and minimal at p), computes the trace of Frobenius $a_p = p + 1 - \#E(\mathbf{F}_p)$. If E is defined over a nonprime finite field $\mathbf{F}_q$, ignore p and return $q + 1 - \#E(\mathbf{F}_q)$. When p is implied (E defined over $\mathbf{Q}_p$ or a finite field), p can be omitted (set to `NULL`).

14.1.7.2 Curves defined a nonprime finite field. In this subsection, we assume that `ell_get_type`(E) is `t_ELL_Fq`. (As noted above, a curve defined over $\mathbf{Z}/p\mathbf{Z}$ can be represented as a `t_ELL_Fq`.)

`GEN FF_elltwist(GEN E)` returns the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of the quadratic twist of E.

`GEN FF_ellmul(GEN E, GEN P, GEN n)` returns $[n]P$ where n is an integer and P is a point on the curve E.

`GEN FF_ellrandom(GEN E)` returns a random point in $E(\mathbf{F}_q)$. This function never returns the point at infinity, unless this is the only point on the curve.

`GEN FF_ellorder(GEN E, GEN P, GEN o)` returns the order of the point P, where o is a multiple of the order of P, or its factorization.

`GEN FF_ellcard(GEN E)` returns $\#E(\mathbf{F}_q)$.

`GEN FF_ellcard_SEA(GEN E, long s)` This function returns $\#E(\mathbf{F}_q)$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$. The parameter s has the same meaning as in `Fp_ellcard_SEA`.

`GEN FF_ellgens(GEN E)` returns the generators of the group $E(\mathbf{F}_q)$.

`GEN FF_elllog(GEN E, GEN P, GEN G, GEN o)` Let `G` be a point of order `o`, return e such that $[e]P = G$. If e does not exists, the result is undefined.

`GEN FF_ellgroup(GEN E, GEN *pm)` returns the structure of the Abelian group $E(\mathbf{F}_q)$ and set `*pm` to m (see `gen_ellgens`).

`GEN FF_ellweilpairing(GEN E, GEN P, GEN Q, GEN m)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN FF_elltatepairing(GEN E, GEN P, GEN Q, GEN m)` returns the Tate pairing of P and Q, where $[m]P = 0$.

14.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

14.2.1 Helper functions.

`GEN elltrace_extension(GEN t, long n, GEN q)` Let E some elliptic curve over $\mathbf{F}_q$ such that the trace of the Frobenius is t, returns the trace of the Frobenius over $\mathbf{F}_q^n$.

14.2.2 Elliptic curves over $\mathbf{F}_p$, $p > 3$.

Let p a prime number and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p$. A `FpE` is a point of $E(\mathbf{F}_p)$. Since an affine point and a_4 determine an unique $a6$, most functions do not take a_6 as an argument. A `FpE` is either the point at infinity (`ellinf()`) or a FpV whith two components. The parameters a_4 and a_6 are given as `t_INT`s when required.

`GEN Fp_ellj(GEN a4, GEN a6, GEN p)` returns the j-invariant of the curve E.

`void Fp_ellj_to_a4a6(GEN j, GEN p, GEN *pa4, GEN *pa6)` sets `*pa4` to a_4 and `*pa6` to a_6 where a_4 and a_6 define a fixed elliptic curve with j-invariant j.

`int Fp_elljissupersingular(GEN j, GEN p)` returns 1 if j is the j-invariant of a supersingular curve over $\mathbf{F}_p$, 0 otherwise.

`GEN Fp_ellcard(GEN a4, GEN a6, GEN p)` returns the cardinality of the group $E(\mathbf{F}_p)$.

`GEN Fp_ellcard_SEA(GEN a4, GEN a6, GEN p, long s)` This function returns $\#E(\mathbf{F}_p)$, using the Schoof-Elkies-Atkin algorithm. If the `seadata` package is installed, the function will be faster.

The extra flag `s`, if set to a nonzero value, causes the computation to return `gen_0` (an impossible cardinality) if one of the small primes ℓ divides the curve order but does not divide s. For cryptographic applications, where one is usually interested in curves of prime order, setting $s = 1$ efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set $s = 2$. If moreover `s` is negative, similar checks are performed for the twist of the curve.

`GEN Fp_ffellcard(GEN a4, GEN a6, GEN q, long n, GEN p)` returns the cardinality of the group $E(\mathbf{F}_q)$ where $q = p^n$.

`GEN Fp_ellgroup(GEN a4, GEN a6, GEN N, GEN p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p)$, which is assumed to be of order N and set `*pm` to m.

`GEN Fp_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN p)` returns generators of the group $E(\mathbf{F}_p)$ with the base change `ch` (see `FpE_changepoint`), where D and m are as returned by `Fp_ellgroup`.

`GEN Fp_elldivpol(GEN a4, GEN a6, long n, GEN p)` returns the n-division polynomial of the elliptic curve E.

`void Fp_elltwist(GEN a4, GEN a6, GEN p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E. Assume p is an odd prime.

14.2.3 `FpE`.

`GEN FpE_add(GEN P, GEN Q, GEN a4, GEN p)` returns the sum $P + Q$ in the group $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$, for any value of a_6 compatible with the points given.

`GEN FpE_sub(GEN P, GEN Q, GEN a4, GEN p)` returns $P - Q$.

`GEN FpE_dbl(GEN P, GEN a4, GEN p)` returns $2.P$.

`GEN FpE_neg(GEN P, GEN p)` returns $-P$.

`GEN FpE_mul(GEN P, GEN n, GEN a4, GEN p)` return $n.P$.

`GEN FpE_changepoint(GEN P, GEN m, GEN a4, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FpV`).

`GEN FpE_changepointinv(GEN P, GEN m, GEN a4, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FpV`).

`GEN random_FpE(GEN a4, GEN a6, GEN p)` returns a random point on $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$.

`GEN FpE_order(GEN P, GEN o, GEN a4, GEN p)` returns the order of P in the group $E(\mathbf{F}_p)$, where o is a multiple of the order of P, or its factorization.

`GEN FpE_log(GEN P, GEN G, GEN o, GEN a4, GEN p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FpE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FpE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN FpE_to_mod(GEN P, GEN p)` returns P as a vector of `t_INTMOD`s.

`GEN RgE_to_FpE(GEN P, GEN p)` returns the `FpE` obtained by applying `Rg_to_Fp` coefficientwise.

`int RgE_is_FpE(GEN P, GEN *p)` check whether `P` can be mapped to a `FpE` using `RgE_to_FpE` by applying `Rg_is_Fp` coeffcient-wise.

14.2.4 `Fle`. Let p be a prime `ulong`, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `ulong`. A `Fle` is either the point at infinity (`ellinf()`), or a `Flv` with two components $[x, y]$.

`long Fl_elltrace(ulong a4, ulong a6, ulong p)` returns the trace t of the Frobenius of $E(\mathbf{F}_p)$. The cardinality of $E(\mathbf{F}_p)$ is thus $p + 1 - t$, which might not fit in an `ulong`.

`long Fl_elltrace_CM(long CM, ulong a4, ulong a6, ulong p)` as `Fl_elltrace`. If `CM` is 0, use the standard algorithm; otherwise assume the curve has CM by a principal imaginary quadratic order of discriminant `CM` and use a faster algorithm. Useful when the curve is the reduction of $E/\mathbf{Q}$, which has CM by a principal order, and we need the trace of Frobenius for many distinct p, see `ellQ_get_CM`.

`ulong Fl_elldisc(ulong a4, ulong a6, ulong p)` returns the discriminant of the curve E.

`ulong Fl_elldisc_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the discriminant of the curve E, assuming pi is the pseudoinverse of p.

`ulong Fl_ellj(ulong a4, ulong a6, ulong p)` returns the j-invariant of the curve E.

`ulong Fl_ellj_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the j-invariant of the curve E, assuming pi is the pseudoinverse of p.

`void Fl_ellj_to_a4a6(ulong j, ulong p, ulong *pa4, ulong *pa6)` sets `*pa4` to a_4 and `*pa6` to a_6 where a_4 and a_6 define a fixed elliptic curve with j-invariant j.

`void Fl_elltwist(ulong a4, ulong a6, ulong p, ulong *pA4, ulong *pA6)` set `*pA4` to A_4 and `*pA6` to A_6 where A_4 and A_6 define the twist of E.

`void Fl_elltwist_disc(ulong a4, ulong a6, ulong D, ulong p, ulong *pA4, ulong *pA6)` sets `*pA4` to A_4 and `*pA6` to A_6 where A_4 and A_6 define the twist of E by the discriminant D.

`GEN Fl_ellptors(ulong l, ulong N, ulong a4, ulong a6, ulong p)` return a basis of the l-torsion subgroup of E.

`GEN Fle_add(GEN P, GEN Q, ulong a4, ulong p)`

`GEN Fle_dbl(GEN P, ulong a4, ulong p)`

`GEN Fle_sub(GEN P, GEN Q, ulong a4, ulong p)`

`GEN Fle_mul(GEN P, GEN n, ulong a4, ulong p)`

`GEN Fle_mulu(GEN P, ulong n, ulong a4, ulong p)`

`GEN Fle_order(GEN P, GEN o, ulong a4, ulong p)`

`GEN Fle_log(GEN P, GEN G, GEN o, ulong a4, ulong p)`

`GEN Fle_tatepairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)`

`GEN Fle_weilpairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)`

`GEN random_Fle(ulong a4, ulong a6, ulong p)`

`GEN random_Fle_pre(ulong a4, ulong a6, ulong p, ulong pi)`

`GEN Fle_changepoint(GEN x, GEN ch, ulong p)`, `ch` is assumed to give the change of coordinates $[u, r, s, t]$ as a `t_VECSMALL`.

`GEN Fle_changepointinv(GEN x, GEN ch, ulong p)`, as `Fle_changepoint`

14.2.5 FpJ.

Let $p > 3$ be a prime `t_INT`, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4 \times x + a_6$, where a_4 and a_6 are `t_INT`. A `FpJ` is a `FpV` with three components $[x, y, z]$, representing the affine point $[x/z^2, y/z^3]$ in Jacobian coordinates, the point at infinity being represented by $[1, 1, 0]$. The following must holds: $y^2 = x^3 + a_4xz^4 + a_6z^6$. For all nonzero u, the points $[u^2x, u^3y, uz]$ and $[x, y, z]$ are representing the same affine point.

`GEN FpJ_add(GEN P, GEN Q, GEN a4, GEN p)`

`GEN FpJ_dbl(GEN P, GEN a4, GEN p)`

`GEN FpJ_mul(GEN P, GEN n, GEN a4, GEN p)`;

`GEN FpJ_neg(GEN P, GEN p)` return $-P$.

`GEN FpJ_to_FpE(GEN P, GEN p)` return the corresponding `FpE`.

`GEN FpE_to_FpJ(GEN P)` return the corresponding `FpJ`.

14.2.6 `Flj`.

Let $p > 3$ be a prime. Below, pi is assumed to be the pseudoinverse of p (see `get_Fl_red`).

`GEN Fle_to_Flj(GEN P)` convert a `Fle` to an equivalent `Flj`.

`GEN Flj_to_Fle(GEN P, ulong p)` convert a `Flj` to the equivalent `Fle`.

`GEN Flj_to_Fle_pre(GEN P, ulong p, ulong pi)` convert a `Flj` to the equivalent `Fle`.

`GEN Flj_add_pre(GEN P, GEN Q, ulong a4, ulong p, ulong pi)`

`GEN Flj_dbl_pre(GEN P, ulong a4, ulong p, ulong pi)`

`GEN Flj_neg(GEN P, ulong p)` return $-P$.

`GEN Flj_mulu_pre(GEN P, ulong n, ulong a4, ulong p, ulong pi)`

`GEN random_Flj_pre(ulong a4, ulong a6, ulong p, ulong pi)`

`GEN Flj_changepointinv_pre(GEN P, GEN ch, ulong p, ulong pi)` where `ch` is the `Flv` $[u, r, s, t]$.

`GEN FljV_factorback_pre(GEN P, GEN L, ulong p, ulong pi)`

14.2.7 Elliptic curves over $\mathbf{F}_{2^n}$. Let T be an irreducible `F2x` and E the elliptic curve given by either the equation $E : y^2 + xy = x^3 + a_2x^2 + a_6$, where a_2, a_6 are `F2x` in $\mathbf{F}_2[X]/(T)$ (ordinary case) or $E : y^2 + a_3y = x^3 + a_4x + a_6$, where a_3, a_4, a_6 are `F2x` in $\mathbf{F}_2[X]/(T)$ (supersingular case).

A `F2xqE` is a point of $E(\mathbf{F}_2[X]/(T))$. In the supersingular case, the parameter `a2` is actually the `t_VEC` $[a_3, a_4, a_3^{-1}]$.

`GEN F2xq_ellcard(GEN a2, GEN a6, GEN T)` Return the order of the group $E(\mathbf{F}_2[X]/(T))$.

`GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pm)` Return the group structure D of the group $E(\mathbf{F}_2[X]/(T))$, which is assumed to be of order N and set `*pm` to m.

`GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T)` Returns generators of the group $E(\mathbf{F}_2[X]/(T))$ with the base change `ch` (see `F2xqE_changepoint`), where D and m are as returned by `F2xq_ellgroup`.

`void F2xq_elltwist(GEN a4, GEN a6, GEN T, GEN *a4t, GEN *a6t)` sets `*a4t` and `*a6t` to the parameters of the quadratic twist of E.

14.2.8 `F2xqE`.

`GEN F2xqE_changepoint(GEN P, GEN m, GEN a2, GEN T)` returns the image Q of the point P on the curve $E : y^2 + x * y = x^3 + a_2x^2 + a_6$ by the coordinate change m (which is a `F2xqV`).

`GEN F2xqE_changepointinv(GEN P, GEN m, GEN a2, GEN T)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `F2xqV`).

`GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_dbl(GEN P, GEN a2, GEN T)`

`GEN F2xqE_neg(GEN P, GEN a2, GEN T)`

`GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)`

`GEN random_F2xqE(GEN a2, GEN a6, GEN T)`

`GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T)` returns the order of P in the group $E(\mathbf{F}_2[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN F2xqE_weilpairing(GEN Q, GEN Q, GEN m, GEN a2, GEN T)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN RgE_to_F2xqE(GEN P, GEN T)` returns the `F2xqE` obtained by applying `Rg_to_F2xq` coefficientwise.

14.2.9 Elliptic curves over $\mathbf{F}_q$, small characteristic $p > 2$. Let $p > 2$ be a prime `ulong`, T an irreducible `Flx` mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `Flx` in $\mathbf{F}_p[X]/(T)$. A `FlxqE` is a point of $E(\mathbf{F}_p[X]/(T))$.

In the special case $p = 3$, ordinary elliptic curves ($j(E) \neq 0$) cannot be represented as above, but admit a model $E : y^2 = x^3 + a_2x^2 + a_6$ with a_2 and a_6 being `Flx` in $\mathbf{F}_3[X]/(T)$. In that case, the parameter `a2` is actually stored as a `t_VEC`, $[a_2]$, to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the j-invariant of the curve E.

`void Flxq_ellj_to_a4a6(GEN j, GEN T, ulong p, GEN *pa4, GEN *pa6)` sets `*pa4` to a_4 and `*pa6` to a_6 where a_4 and a_6 define a fixed elliptic curve with j-invariant j.

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and sets `*pm` to m.

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FlxqE_changepoint`), where D and m are as returned by `Flxq_ellgroup`.

`void Flxq_elltwist(GEN a4, GEN a6, GEN T, ulong p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E.

14.2.10 `FlxqE`.

Let $p > 2$ be a prime number.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

GEN FlxqE_neg(GEN P, GEN T, ulong p)

GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)

GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)

GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p) returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p) Let G be a point of order o, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p) returns the Tate pairing of the point of m-torsion P and the point Q.

GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p) returns the Weil pairing of the points of m-torsion P and Q.

GEN FlxqE_weilpairing_pre(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p, ulong pi) , where pi is a pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN RgE_to_FlxqE(GEN P, GEN T, ulong p) returns the FlxqE obtained by applying Rg_to_Flxq coefficientwise.

GEN Flxq_elldivpolmod(GEN a4, GEN a6, long n, GEN h, GEN T, ulong p) returns the n-division polynomial of the elliptic curve E modulo the polynomial h.

14.2.11 Elliptic curves over $\mathbf{F}_q$, large characteristic .

Let $p > 3$ be a prime number, T an irreducible polynomial mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$ with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$. A FpXQE is a point of $E(\mathbf{F}_p[X]/(T))$.

GEN FpXQ_ellj(GEN a4, GEN a6, GEN T, GEN p) returns the j-invariant of the curve E.

int FpXQ_elljissupersingular(GEN j, GEN T, GEN p) returns 1 if j is the j-invariant of a supersingular curve over $\mathbf{F}_p[X]/(T)$, 0 otherwise.

int Fq_elljissupersingular(GEN j, GEN T, GEN p) as FpXQ_elljissupersingular but j can be a t_INT.

GEN ellsupersingularj_FpXQ(GEN T, GEN p) T being a ZX of degree 2, return a random supersingular j-invariant in $\mathbf{F}_p[X]/T$.

GEN FpXQ_ellcard(GEN a4, GEN a6, GEN T, GEN p) returns the order of $E(\mathbf{F}_p[X]/(T))$.

GEN FpXQ_ellcard_supersingular(GEN a4, GEN a6, GEN T, GEN p) This function returns $\#E(\mathbf{F}_p[X]/(T))$, assuming the curve is supersingular, see FpXQ_elljissupersingular.

GEN Fq_ellcard_supersingular(GEN a4, GEN a6, GEN T, GEN p) This function is identical to FpXQ_ellcard_supersingular, except that it allows T to be NULL.

GEN Fq_ellcard_SEA(GEN a4, GEN a6, GEN q, GEN T, GEN p, long s) This function returns $\#E(\mathbf{F}_p[X]/(T))$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$, and q is the cardinality of $\mathbf{F}_p[X]/(T)$. The parameter s has the same meaning as in Fp_ellcard_SEA. If the seadata package is installed, the function will be faster.

GEN FpXQV_ellcharpoly(GEN E, GEN T, GEN p) Let E be a vector $[a_1, a_2, a_3, a_4, a_6]$ defining the plane curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ over $\mathbf{F}_p[X]/(T)$, returns the characteristic

polynomial of the Frobenius acting on its first l-adic cohomology group (which can be of degree 0, 1 or 2.)

`GEN FpXQ_ellgroup(GEN a4, GEN a6, GEN N, GEN T, GEN p, GEN *pm)` Return the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and set `*pm` to m.

`GEN FpXQ_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, GEN p)` Returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FpXQE_changepoint`), where D and m are as returned by `FpXQ_ellgroup`.

`GEN FpXQ_elldivpol(GEN a4, GEN a6, long n, GEN T, GEN p)` returns the n-division polynomial of the elliptic curve E.

`GEN Fq_elldivpolmod(GEN a4, GEN a6, long n, GEN h, GEN T, GEN p)` returns the n-division polynomial of the elliptic curve E modulo the polynomial h.

`void FpXQ_elltwist(GEN a4, GEN a6, GEN T, GEN p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E.

14.2.12 `FpXQE`.

`GEN FpXQE_changepoint(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FpXQV`).

`GEN FpXQE_changepointinv(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FpXQV`).

`GEN FpXQE_add(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_sub(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_dbl(GEN P, GEN a4, GEN T, GEN p)`

`GEN FpXQE_neg(GEN P, GEN T, GEN p)`

`GEN FpXQE_mul(GEN P, GEN n, GEN a4, GEN T, GEN p)`

`GEN random_FpXQE(GEN a4, GEN a6, GEN T, GEN p)`

`GEN FpXQE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, GEN p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FpXQE_order(GEN P, GEN o, GEN a4, GEN T, GEN p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN FpXQE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FpXQE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN RgE_to_FpXQE(GEN P, GEN T, GEN p)` returns the `FpXQE` obtained by applying `Rg_to_FpXQ` coefficientwise.

`int RgE_is_FpXQE(GEN P, GEN *T, GEN *p)` check whether `P` can be mapped to a `FpXQE` using `RgE_to_FpXQE` by applying `Rg_is_FpXQ` coeffcient-wise.

14.3 Functions related to modular polynomials.

Variants of `polmodular`, returning the modular polynomial of prime level L for the invariant coded by `inv` (0: j, 1: Weber-f, see `polclass` for the full list).

`GEN polmodular_ZXX(long L, long inv, long vx, long vy)` returns a bivariate polynomial in variables `vx` and `vy`.

`GEN polmodular_ZM(long L, long inv)` returns a matrix of (integral) coefficients.

`GEN Fp_polmodular_evalx(long L, long inv, GEN J, GEN p, long v, int derivs)` returns the modular polynomial evaluated at J modulo the prime p in the variable v (if `derivs` is nonzero, returns a vector containing the modular polynomial and its first and second derivatives, all evaluated at J modulo p).

14.3.1 Functions related to modular invariants.

`void check_modinv(long inv)` report an error if `inv` is not a valid code for a mdular invariant.

`int modinv_good_disc(long inv, long D)` test whether the invariant `inv` is defined for the discriminant `D`.

`int modinv_good_prime(long inv, long D)` test whether the invariant `inv` is defined for the prime `p`.

`long modinv_height_factor(long inv)` return the height factor of the modular invariant `inv` with respect to the j-invariant. This is an integer n such that the j-invariant is asymptotically of the order of the n-th power of the invariant `inv`.

`long modinv_is_Weber(long inv)` test whether the invariant `inv` is a power of Weber f.

`long modinv_is_double_eta(long inv)` test whether the invariant `inv` is a double η quotient.

`long disc_best_modinv(long D)` the integer D being a negative discriminant, return the modular invariant compatible with D with the highest height factor.

`GEN Fp_modinv_to_j(GEN x, long inv, GEN p)` Let Φ the modular equation between j and the modular invariant `inv`, return y such that $\Phi(y, x) = 0 \pmod p$.

14.4 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space $\mathbf{P}_{(1,d,1)}$, with coordinates (x, y, z) and a model of the form $y^2 = T(x, z)$, where T is homogeneous of degree $2d$, and squarefree. Thus the curve is nonsingular of genus $d - 1$.

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that $T \in \mathbf{Z}[X]$ is integral. Returns 1 if the curve is locally soluble over $\mathbf{Q}_p$, 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let K be a number field, attached to `nf`, `pr` a *prid* attached to some maximal ideal `p`; assumes that $T \in \mathbf{Z}_K[X]$ is integral. Returns 1 if the curve is locally soluble over $K_{\mathtt{p}}$. The argument `nf` is a true *nf* structure.

Chapter 15:
L-functions

15.1 Accessors.

```
long is_linit(GEN data)
GEN ldata_get_an(GEN ldata)
GEN ldata_get_dual(GEN ldata)
long ldata_isreal(GEN ldata)
GEN ldata_get_gammavec(GEN ldata)
long ldata_get_degree(GEN ldata)
GEN ldata_get_k(GEN ldata)
GEN ldata_get_k1(GEN ldata)
GEN ldata_get_conductor(GEN ldata)
GEN ldata_get_rootno(GEN ldata)
GEN ldata_get_residue(GEN ldata)
long ldata_get_type(GEN ldata)
long linit_get_type(GEN linit)
GEN linit_get_ldata(GEN linit)
GEN linit_get_tech(GEN linit)
GEN lfun_get_domain(GEN tech)
GEN lfun_get_dom(GEN tech)
long lfun_get_bitprec(GEN tech)
GEN lfun_get_factgammavec(GEN tech)
GEN lfun_get_step(GEN tech)
GEN lfun_get_pol(GEN tech)
GEN lfun_get_Residue(GEN tech)
GEN lfun_get_k2(GEN tech)
GEN lfun_get_w2(GEN tech)
GEN lfun_get_expot(GEN tech)
long lfun_get_bitprec(GEN tech)
```

`GEN lfunprod_get_fact(GEN tech)`

`GEN theta_get_an(GEN tdata)`

`GEN theta_get_K(GEN tdata)`

`GEN theta_get_R(GEN tdata)`

`long theta_get_bitprec(GEN tdata)`

`long theta_get_m(GEN tdata)`

`GEN theta_get_tdom(GEN tdata)`

`GEN theta_get_isqrtN(GEN tdata)`

15.2 Conversions and constructors.

`GEN lfunmisc_to_ldata(GEN obj)` converts `obj` to `Ldata` format. Exception if `obj` cannot be converted.

`GEN lfunmisc_to_ldata_shallow(GEN obj)` as `lfunmisc_to_ldata`, shallow result. Exception if `obj` cannot be converted.

`GEN lfunmisc_to_ldata_shallow_i(GEN obj)` as `lfunmisc_to_ldata_shallow`, returning `NULL` on failure.

`GEN lfunrtopoles(GEN r)` given a vector r of "polar parts" $[\beta_i, P_i]$ where β_i is a complex number and P_i is a power series return the (β_i), sorted with respect to `cmp_universal`.

`int sdomain_isincl(double k, GEN dom, GEN dom0)`

`GEN ldata_vecan(GEN ldata, long N, long prec)` return the vector of coefficients of indices 1 to N to precision `prec`. The output is allowed to be a `t_VECSMALL` when the coefficients are known to be all integral and fit into a `long`; for instance the Dirichlet L function of a real character or the L-function of a rational elliptic curve.

`GEN ldata_newprec(GEN ldata, long prec)` return a shallow copy of `ldata` with fields accurate to precision `prec`.

`long etaquotype(GEN *peta, GEN *pN, GEN *pk, GEN *pCHI, long *pv, long *psd, long *pcusp)` Let `eta` be the integer matrix factorization supposedly attached to an η-quotient $f(z) = \prod_i \eta(n_i z)^{e_i}$. Assuming `*peta` is initially set to `eta`, this function returns 0 if there is a type error or this does not define a function on some $X_0(N)$. Else it returns 1 and sets

- `*peta` to a normalized factorization (as would be returned by `factor`),
- `*pN` to the level N of f,
- `*pk` to the modular weight k of f,
- `*pCHI` to the Nebentypus of f (quadratic character) as an integer,
- `*pv` to the valuation at infinity $v_q(f)$,
- `*psd` to 1 if and only if f is self-dual,
- `*pcusp` to 1 if f is cuspidal, else to 0 if f holomorphic at all cusps, else to -1.

The last three arguments `pCHI`, `pv` and `pcusp` can be set to `NULL`, in which case the relevant information is not computed, which saves time.

15.3 Variants of GP functions.

`GEN lfun(GEN ldata, GEN s, long bitprec)`

`GEN lfuninit(GEN ldata, GEN dom, long der, long bitprec)`

`GEN lfuninit_make(long t, GEN ldata, GEN tech, GEN domain)`

`GEN lfunlambda(GEN ldata, GEN s, long bitprec)`

`GEN lfunquadneg(long D, long k)` for $L(\chi_D, k)$, D fundamental discriminant and $k \geq 0$.

`long lfunthetacost(GEN ldata, GEN tdom, long m, long bit, long *extrabit)`: `lfunthetacost0` when the first argument is known to be an `Ldata`. The `*extrabit` value is set to an upper estimate for the maximum over n of the exponent of the m-th derivative of $a_n K(nt/\sqrt{N})$, for t in the `tdom` domain. For an absolute error $2^{-\texttt{bit}}$, the values summed in θ should be computed at accuracy `bit` + $*$`extrabit`.

`GEN lfunthetacheckinit(GEN data, GEN tinf, long m, long bitprec)`

`GEN lfunrootno(GEN data, long bitprec)`

`GEN lfunabelianrelinit(GEN bnr, GEN subg, GEN dom, long der, long bitprec)` where `bnr` is a true *bnr* structure and `subg` is a congruence subgroup. Not GC-clean.

`GEN lfunzetakinit(GEN nf, GEN dom, long der, long bitprec)` where `nf` is a true *nf* structure. Not GC-clean.

`GEN lfunellmfpeters(GEN E, long bitprec)`

15.4 Inverse Mellin transforms of Gamma products.

`GEN gammamellininv(GEN Vga, GEN s, long m, long bitprec)`

`GEN gammamellininvinit(GEN Vga, long m, long bitprec)`

`GEN gammamellininvrt(GEN K, GEN s, long bitprec)` not stack-clean, but suitable for `gc_upto`.

`int Vgaeasytheta(GEN Vga)` return 1 if the inverse Mellin transform is an exponential and 0 otherwise.

`double dbllemma526(double a, double b, double c, long B)`

`double dblcoro526(double a, double c, long B)`

Chapter 16:
Modular symbols

`void checkms(GEN W)` raise an exception if W is not an *ms* structure from `msinit`.

`void checkmspadic(GEN W)` raise an exception if W is not an *mspadic* structure from `mspadicinit`.

`GEN mseval2_ooQ(GEN W, GEN phi, GEN c)` let W be a `msinit` structure for $k = 2$, ϕ be a modular symbol with integral values and c be a rational number. Return the integer $\phi(p)$, where p is the path $\{\infty, c\}$.

`void mspadic_parse_chi(GEN s, GEN *s1, GEN *s2)` see `mspadicL`; let χ be the cyclotomic character from $\mathrm{Gal}\big(\mathbf{Q}_p(mu_{p^\infty})/\mathbf{Q}_p\big)$ to $\mathbf{Z}_p^*$ and τ be the Teichmüller character for $p > 2$ and the character of order 2 on $(\mathbf{Z}/4\mathbf{Z})^*$ if $p = 2$. Let s encode the p-adic characther $\chi^s := \langle\chi\rangle^{s_1}\tau^{s_2}$; set `*s1` and `*s2` to the integers s_1 and s_2.

`GEN mspadic_unit_eigenvalue(GEN ap, long k, GEN p, long n)` let p be a prime not dividing the trace of Frobenius `ap`, return the unit root of $x^2 - ap * x + p^(k-1)$ to p-adic accuracy p^n.

Variants of `mfnumcusps` :

`ulong mfnumcuspsu(ulong n)`

`GEN mfnumcusps_fact(GEN fa)` where `fa` is `factor`(n).

`ulong mfnumcuspsu_fact(GEN fa)` where `fa` is `factoru`(n).

Chapter 17:
Modular forms

17.1 Implementation of public data structures.

`void checkMF(GEN mf)` raise an exception if the argument is not a modular form space.

`GEN checkMF_i(GEN mf)` return the underlying modular form space if `mf` is either directly a modular form space from `mfinit` or a symbol from `mfsymbol`. Return `NULL` otherwise.

`int checkmf_i(GEN mf)` return 1 if the argument is a modular form and 0 otherwise.

`int checkfarey_i(GEN F)` return 1 if the argument is a Farey symbol (from `mspolygon` or `msfarey`) and 0 otherwise.

17.1.1 Accessors for modular form spaces.

Shallow functions; assume that their argument is a modular form space is created by `mfinit` and checked using `checkMF`.

`GEN MF_get_gN(GEN mf)` return the level N as a `t_INT`.

`long MF_get_N(GEN mf)` return the level N as a `long`.

`GEN MF_get_gk(GEN mf)` return the level k as a `t_INT`.

`long MF_get_k(GEN mf)` return the level k as a `long`.

`long MF_get_r(GEN mf)` assuming the level is a half-integer, return the integer $r = k - (1/2)$.

`GEN MF_get_CHI(GEN mf)` return the nebentypus χ, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or `t_POLMOD` in t.

`long MF_get_space(GEN mf)` returns the space type, corresponding to `mfinit`'s `space` flag. The current list is

```
mf_NEW, mf_CUSP, mf_OLD, mf_EISEN, mf_FULL
```

`GEN MF_get_basis(GEN mf)` return the $\mathbf{Q}$-basis of the space, concatenation of `MF_get_E` and `MF_get_S`, in this order; the forms have coefficients in $\mathbf{Q}(\chi)$. Low-level version of `mfbasis`.

`long MF_get_dim(GEN mf)` returns the dimension d of the space. It is the cardinality of `MF_get_basis`.

`GEN MF_get_E(GEN mf)` returns a $\mathbf{Q}$-basis for the subspace spanned by Eisenstein series in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

`GEN MF_get_S(GEN mf)` returns a $\mathbf{Q}$-basis for the cuspidal subspace in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

`GEN MF_get_fields(GEN mf)` returns the vector of polynomials defining each Galois orbit of newforms over $\mathbf{Q}(\chi)$. Uses memoization: a first call splits the space and may be costly; subsequent calls return the cached result.

`GEN MF_get_newforms(GEN mf)` returns a vector `vF` containing the coordinates of the eigenforms on `MF_get_basis` (`mftobasis` form). Low-level version of `mfeigenbasis`, whose elements are recovered as `mflinear(mf, gel(vF,i))`. Uses memoization, sharing the same data as `MF_get_fields`. Note that it is much more efficient to use `mfcoefs(mf,)` then multiply by this vector than to compute the coefficients of eigenforms from `mfeigenbasis` individually.

The following accessors are technical,

`GEN MF_get_M(GEN mf)` the $(1+m)\times d$ matrix whose j-th column contain the coefficients of the j-th entry in `MF_get_basis`, m is the optimal "Sturm bound" for the space: the maximum of the $v_\infty(f)$ over nonzero forms. It has entries in $\mathbf{Q}(\chi)$.

`GEN MF_get_Mindex(GEN mf)` is a `t_VECSMALL` containing d row indices, the corresponding rows of M form an invertible matrix M_0.

`GEN MF_get_Minv(GEN mf)` the inverse of M_0 in a form suitable for fast multiplication.

`GEN MFcusp_get_vMjd(GEN mf)` valid only for a full *cuspidal* space. Then the functions in `MF_get_S` are of the form $B_d T_j Tr_M^{new}$. This returns the vector of triples (`t_VECSMALL`) $[M,j,d]$, in the same order.

`GEN MFnew_get_vj(GEN mf)` valid only for a *new* space. Then the functions in `MF_get_S` are of the form $T_j Tr_N^{new}$. This returns a `t_VECSMALL` of the Hecke indices j, in the same order.

17.1.2 Accessors for individual modular forms.

`GEN mf_get_gN(GEN F)` return the level of F, which may be a multiple of the conductor, as a `t_INT`

`long mf_get_N(GEN F)` return the level as a `long`.

`GEN mf_get_gk(GEN F)` return the weight of F as a `t_INT` or a `t_FRAC` with denominator 2 (half-integral weight).

`long mf_get_k(GEN F)` return the weight as a `long`; if the weight is not integral, this raises an exception.

`long mf_get_r(GEN F)` assuming F is a modular form of half-integral weight $k=(2r+1)/2$, return $r=k-(1/2)$.

`GEN mf_get_CHI(GEN F)` return the nebentypus, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or `t_POLMOD` in t.

`GEN mf_get_field(GEN F)` return the polynomial (in variable y) defining $\mathbf{Q}(f)$ over $\mathbf{Q}(\chi)$.

`GEN mf_get_NK(GEN F)` return the tag attached to F: a vector containing `gN`, `gk`, `CHI`, `field`. Never use its component directly, use individual accessors as above.

`long mf_get_type(GEN F)` returns a symbolic name for the constructor used to create the form, e.g. `t_MF_EISEN` for a general Eisenstein series. A form has a recursive structure represented by a tree: its definition may involve other forms, e.g. the tree attached to $T_n f$ contains f as a subtree. Such trees have *leaves*, forms which do not contain a strict subtree, e.g. `t_MF_DELTA` is a leaf, attached to Ramanujan's Δ.

Here is the current list of types; since the names are liable to change, they are not documented at this point. Use `mfdescribe` to visualize their mathematical structure.

```
/*leaves*/
  t_MF_CONST, t_MF_EISEN, t_MF_Ek, t_MF_DELTA, t_MF_ETAQUO, t_MF_ELL,
  t_MF_DIHEDRAL, t_MF_THETA, t_MF_TRACE, t_MF_NEWTRACE,
/*recursive*/
  t_MF_MUL, t_MF_POW, t_MF_DIV, t_MF_BRACKET, t_MF_LINEAR, t_MF_LINEAR_BHN,
  t_MF_SHIFT, t_MF_DERIV, t_MF_DERIVE2, t_MF_TWIST, t_MF_HECKE,
  t_MF_BD,
```

17.1.3 Nebentypus. The characters stored in modular forms and modular form spaces have a special structure. One can recover the parameters of an ordinary Dirichlet character by `G = gel(CHI,1)` (the underlying `znstar`) and `chi = gel(CHI,2)` (the underlying character in `znconreylog` form).

`long mfcharmodulus(GEN CHI)` the modulus of χ.

`long mfcharorder(GEN CHI)` the order of χ.

`GEN mfcharpol(GEN CHI)` the cyclotomic polynomial Φ_n defining $\mathbf{Q}(\chi)$, always normalized so that n is not 2 mod 4.

17.1.4 Miscellaneous functions.

`long mfnewdim(long N, long k, GEN CHI)` dimension of the new part of the cuspidal space.

`long mfcuspdim(long N, long k, GEN CHI)` dimension of the cuspidal space.

`long mfolddim(long N, long k, GEN CHI)` dimension of the old part of the cuspidal space.

`long mfeisensteindim(long N, long k, GEN CHI)` dimension of the Eisenstein subspace.

`long mffulldim(long N, long k, GEN CHI)` dimension of the full space.

`GEN mfeisensteinspaceinit(GEN NK)`

`GEN mfdiv_val(GEN F, GEN G, long vG)`

`GEN mfembed(GEN E, GEN v)`

`GEN mfmatembed(GEN E, GEN v)`

`GEN mfvecembed(GEN E, GEN v)`

`long mfsturmNgk(long N, GEN k)`

`long mfsturmNk(long N, long k)`

`long mfsturm_mf(GEN mf)`

`long mfiscuspidal(GEN mf, GEN F)`

`GEN mftobasisES(GEN mf, GEN F)`

`GEN mftocol(GEN F, long lim, long d)`

`GEN mfvectomat(GEN vF, long lim, long d)`

Chapter 18:
Plots

A `PARI_plot` canvas is a record of dimensions, with the following fields:

```
long width;  /* window width */
long height; /* window height */
long hunit;  /* length of horizontal 'ticks' */
long vunit;  /* length of vertical 'ticks' */
long fwidth; /* font width */
long fheight;/* font height */
void (*draw)(PARI_plot *T, GEN w, GEN x, GEN y);
```

The `draw` method performs the actual drawing of a `t_VECSMALL` w (rectwindow indices); x and y are `t_VECSMALL`s of the same length and rectwindow $w[i]$ is drawn with its upper left corner at offset $(x[i], y[i])$. No plot engine is available in `libpari` by default, since thie would introduce a dependency on extra graphical libraries. See the files `src/graph/plot*` for basic implementations of various plot engines: `plotsvg` is particularly simple (`draw` is a 1-liner).

`void pari_set_plot_engine(void (*T)(PARI_plot *))` installs the graphical engine T and initializes the graphical subsystem. No routine in this chapter will work without this initialization.

`void pari_kill_plot_engine(void)` closes the graphical subsystem and frees the ressources it occupies.

18.1 Highlevel functions.

Those functions plot $f(E, x)$ for $x \in [a, b]$, using n regularly spaced points (by default).

`GEN ploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw physically.

`GEN plotrecth(void *E, GEN(*f)(void*, GEN), long w, GEN a, GEN b, long flags, long n, long prec)` draw in rectwindow w.

18.2 Function.

`void plotbox(long ne, GEN gx2, GEN gy2)`

`void plotclip(long rect)`

`void plotcolor(long ne, long color)`

`void plotcopy(long source, long dest, GEN xoff, GEN yoff, long flag)`

`GEN plotcursor(long ne)`

`void plotdraw(GEN list, long flag)`

`GEN plothraw(GEN listx, GEN listy, long flag)`

`GEN plothsizes(long flag)`

`void plotinit(long ne, GEN x, GEN y, long flag)`

`void plotkill(long ne)`

`void plotline(long ne, GEN x2, GEN y2)`

`void plotlines(long ne, GEN listx, GEN listy, long flag)`

`void plotlinetype(long ne, long t)`

`void plotmove(long ne, GEN x, GEN y)`

`void plotpoints(long ne, GEN listx, GEN listy)`

`void plotpointsize(long ne, GEN size)`

`void plotpointtype(long ne, long t)`

`void plotrbox(long ne, GEN x2, GEN y2)`

`GEN plotrecthraw(long ne, GEN data, long flags)`

`void plotrline(long ne, GEN x2, GEN y2)`

`void plotrmove(long ne, GEN x, GEN y)`

`void plotrpoint(long ne, GEN x, GEN y)`

`void plotscale(long ne, GEN x1, GEN x2, GEN y1, GEN y2)`

`void plotstring(long ne, char *x, long dir)`

18.2.1 Obsolete functions. These draw directly to a PostScript file specified by a global variable and should no longer be used. Use `plotexport` and friends instead.

`void psdraw(GEN list, long flag)`

`GEN psplothraw(GEN listx, GEN listy, long flag)`

`GEN psploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw to a PostScript file.

18.3 Dump rectwindows to a PostScript or SVG file.

w, x, y are three `t_VECSMALL`s indicating the rectwindows to dump, at which offsets. If T is `NULL`, rescale with respect to the installed graphic engine dimensions; else with respect to T.

`char* rect2ps(GEN w, GEN x, GEN y, PARI_plot *T)`

`char* rect2ps_i(GEN w, GEN x, GEN y, PARI_plot *T, int plotps)` if `plotps` is 0, as above; else private version used to implement the `plotps` graphic engine (do not rescale, rotate to portrait orientation).

`char* rect2svg(GEN w, GEN x, GEN y, PARI_plot *T)`

18.4 Technical functions exported for convenience.

`void pari_plot_by_file(const char *env, const char *suf, const char *img)` backend used by the `plotps` and `plotsvg` graphic engines.

`void colorname_to_rgb(const char *s, int *r, int *g, int *b)` convert an `X11` colorname to RGB values.

`void color_to_rgb(GEN c, int *r, int *g, int *b)` convert a pari color (`t_VECSMALL` RGB triple or `t_STR` name) to RGB values.

`void long_to_rgb(long c, int *r, int *g, int *b)` split a standard hexadecimal color value `0xfdf5e6` to its rgb components (`0xfd`, `0xf5`, `0xe6`).

Appendix A:
A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions.

```
CC = cc
INCDIR = /usr/include/x86_64-linux-gnu
LIBDIR = /usr/lib/x86_64-linux-gnu
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all:  extgcd

extgcd:  extgcd.c
      $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section 4.10.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;

  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;

    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  return gc_all(av, 3, &a, U, V);
}

int
```

```
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

Appendix B:
PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `__thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
  GEN data = pari_thread_start((struct pari_thread*)arg);
  GEN result = ...; /* Compute result from data */
  pari_thread_close();
  return (void*)result;
}
```

Parent thread:

```
  pthread_t th;
  struct pari_thread pth;
  GEN data, result;

  pari_thread_alloc(&pth, s, data);
  pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
  ... /* do stuff in parent */
  pthread_join(th, (void*)&result); /* wait until child terminates */
  result = gcopy(result); /* copy result from thread stack to main stack */
  pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_valloc(struct pari_thread *pth, size_t s, size_t v, GEN arg)` Allocate a PARI stack of size `s` which can grow to at most `v` (as with `parisize` and `parisizemax`) and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` As above but the stack cannot grow beyond `s`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack attached to the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any `GEN` objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

`GEN pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

`void pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a `GEN` returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenable it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h>   /* Include POSIX threads headers */

#ifndef ENABLE_TLS
#error "this example requires Configure --enable-tls or --mt=ptread"
#endif

void *
mydet(void *arg)
{
  GEN F, M;
  /* Set up thread stack and get thread parameter */
  M = pari_thread_start((struct pari_thread*) arg);
  F = QM_det(M);
  /* Free memory used by the thread */
  pari_thread_close();
  return (void*)F;
}

void *
myfactor(void *arg)  /* same principle */
{
  GEN F, N;
  N = pari_thread_start((struct pari_thread*) arg);
  F = factor(N);
  pari_thread_close();
  return (void*)F;
}

int
main(void)
{
```

```
  long prec = DEFAULTPREC;
  GEN M1,M2, N1,N2, F1,F2, D1,D2;
  pthread_t th1, th2, th3, th4; /* POSIX-thread variables */
  struct pari_thread pth1, pth2, pth3, pth4; /* pari thread variables */

  /* Initialise the main PARI stack and global objects (gen_0, etc.) */
  pari_init(32000000,500000);
  /* Compute in the main PARI stack */
  N1 = addis(int2n(256), 1); /* 2^256 + 1 */
  N2 = subis(int2n(193), 1); /* 2^193 - 1 */
  M1 = mathilbert(149);
  M2 = mathilbert(150);
  /* Allocate pari thread structures */
  pari_thread_alloc(&pth1,8000000,N1);
  pari_thread_alloc(&pth2,8000000,N2);
  pari_thread_alloc(&pth3,32000000,M1);
  pari_thread_alloc(&pth4,32000000,M2);
  /* pthread_create() and pthread_join() are standard POSIX-thread
   * functions to start and get the result of threads. */
  pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
  pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
  pthread_create(&th3,NULL, &mydet,    (void*)&pth3);
  pthread_create(&th4,NULL, &mydet,    (void*)&pth4); /* Start 4 threads */
  pthread_join(th1,(void*)&F1);
  pthread_join(th2,(void*)&F2);
  pthread_join(th3,(void*)&D1);
  pthread_join(th4,(void*)&D2); /* Wait for termination, get the results */
  pari_printf("F1=%Ps\nF2=%Ps\nlog(D1)=%Ps\nlog(D2)=%Ps\n",
              F1,F2, glog(D1,prec),glog(D2,prec));
  pari_thread_free(&pth1);
  pari_thread_free(&pth2);
  pari_thread_free(&pth3);
  pari_thread_free(&pth4); /* clean up */
  return 0;
}
```

Index

SomeWord refers to PARI-GP concepts.
`SomeWord` is a PARI-GP keyword.
SomeWord is a generic index entry.

A

`ABC_to_bnr` . . . 329
`abelian_group` . . . 264
`abgrp_get_cyc` . . . 300
`abgrp_get_gen` . . . 300
`abgrp_get_no` . . . 300
`abmap_char_image` . . . 329
`abmap_kernel` . . . 329
`abmap_nchar_image` . . . 329
`abmap_subgroup_image` . . . 329
`abscmpii` . . . 94
`abscmpiu` . . . 94
`abscmprr` . . . 94
`abscmpui` . . . 94
`absdiviu_rem` . . . 99
`absequalii` . . . 94
`absequaliu` . . . 94
`absequalui` . . . 94
`absfrac` . . . 249
`absfrac_shallow` . . . 249
`absi` . . . 92
`absi_shallow` . . . 93
`absr` . . . 92
`absrnz_equal1` . . . 94
`absrnz_equal2n` . . . 94
`abstorel` . . . 330
`absZ_factor` . . . 175
`absZ_factor_limit` . . . 176
`absZ_factor_limit_strict` . . . 176
addhelp . . . 78
`addii` . . . 15
`addii_sign` . . . 97
`addir` . . . 15
`addir_sign` . . . 97
`addis` . . . 15
`addiu` . . . 96
`addll` . . . 81
`addllx` . . . 81
`addmul` . . . 81
`addmulii` . . . 96
`addmulii_inplace` . . . 96
`addmuliu` . . . 96
`addmuliu_inplace` . . . 97
`addri` . . . 15
`addrr` . . . 15
`addrr_sign` . . . 97
`addsi_sign` . . . 97
`addui` . . . 96
`addui_sign` . . . 97
`addumului` . . . 96
`adduu` . . . 96
`affc_fixlg` . . . 261
`affectsign` . . . 63
`affectsign_safe` . . . 63
`affgc` . . . 261
`affgr` . . . 87
`affii` . . . 87
`affir` . . . 87
`affiz` . . . 87
`affrr` . . . 87
`affrr_fixlg` . . . 87
`affsi` . . . 87
`affsr` . . . 87
`affsz` . . . 87
`affui` . . . 87
`affur` . . . 87
`alarm` . . . 277
`alginit` . . . 349
`alglat_get_primbasis` . . . 350
`alglat_get_scalar` . . . 350
`algmodpr` . . . 350
`algmodprinit` . . . 350
`algmodpr_get_ff` . . . 350
`algmodpr_get_k` . . . 350
`algmodpr_get_lift` . . . 350
`algmodpr_get_m` . . . 350
`algmodpr_get_p` . . . 351
`algmodpr_get_pr` . . . 350
`algmodpr_get_proj` . . . 350
`algmodpr_get_T` . . . 351
`algmodpr_get_tau` . . . 351
`algnewprec` . . . 351
`algnewprec_shallow` . . . 351
`algsimpledec_ss` . . . 351
`algtype` . . . 349
`alg_complete` . . . 348
`alg_csa_table` . . . 348
`alg_cyclic` . . . 348
`alg_get_absdim` . . . 349
`alg_get_abssplitting` . . . 350
`alg_get_aut` . . . 349
`alg_get_auts` . . . 350
`alg_get_b` . . . 350

alg_get_basis 350
alg_get_center 350
alg_get_char 350
alg_get_degree 349
alg_get_dim 349
alg_get_hasse_f 350
alg_get_hasse_i 350
alg_get_invbasis 350
alg_get_invol 350
alg_get_multable 350
alg_get_relmultable 350
alg_get_splitpol 350
alg_get_splittingbasis 350
alg_get_splittingbasisinv 350
alg_get_splittingdata 350
alg_get_splittingfield 350
alg_get_tracebasis 350
alg_hasse 348
alg_hilbert 348
alg_matrix 348
alg_model 349
alg_type 349
atanhui 260
atanhuu 259
avma 17

B

bb_algebra 219
bb_field 218
bb_group 215
bb_ring 220
bernfrac 262, 263
Bernoulli 262, 263
bernreal 262, 263
bezout 101
bfffo 81
bid_get_arch 304
bid_get_archp 305
bid_get_cyc 305
bid_get_fact 305
bid_get_fact2 305
bid_get_gen 305
bid_get_gen_nocheck 305
bid_get_grp 304
bid_get_ideal 304, 305
bid_get_mod 304
bid_get_MOD 304
bid_get_no 305
bid_get_sarch 305
bid_get_sprk 305
bid_get_U 305
BIGDEFAULTPREC 16, 64
bigomegau 106
BIL 51
binary quadratic form 31
binary_2k 90, 91
binary_2k_nv 90
binary_zv 90
bincopy_relink 68
binomial 249
binomialuu 249
bin_copy 68
bitprecision0 224
BITS_IN_HALFULONG 64
BITS_IN_LONG 51, 64, 90, 91
bits_to_int 90
bits_to_u 90
bit_accuracy 59
bit_accuracy_mul 59
bit_prec 59
bl_base 73
bl_next 73
bl_num 73
bl_prev 73
bl_refc 73
bnfgwgeneric 330
bnfisprincipal0 . . . . . 307, 323, 326
bnfisunit 313
bnfnewprec 307, 324
bnfnewprec_shallow 307
bnftestprimes 324
bnf_build_cheapfu 306
bnf_build_cycgen 306
bnf_build_matalpha 306
bnf_build_units 306
bnf_compactfu 303
bnf_compactfu_mat 303
bnf_get_clgp 302
bnf_get_cyc 302
bnf_get_fu 302
bnf_get_fu_nocheck 302
bnf_get_gen 302
bnf_get_logfu 302
bnf_get_nf 302
bnf_get_no 302
bnf_get_reg 302
bnf_get_sunits 302

bnf_get_tuN . . . 302
bnf_get_tuU . . . 302
bnf_has_fu . . . 303
bnrclassno . . . 327
bnrconductor . . . 328
bnrconductorofchar . . . 333
bnrconductor_factored . . . 328
bnrconductor_factored_arch . . . 328
bnrconductor_raw . . . 328
bnrdisc . . . 329
bnrdisclist0 . . . 333
bnrinit0 . . . 332
bnrisconductor . . . 329
bnrisprincipal . . . 333
bnrnewprec . . . 307
bnrnewprec_shallow . . . 307
bnrsurjection . . . 329
bnrtoprimitive . . . 328
bnr_char_sanitize . . . 328
bnr_get_bid . . . 303
bnr_get_bnf . . . 303
bnr_get_clgp . . . 303
bnr_get_cyc . . . 303, 329
bnr_get_gen . . . 303
bnr_get_gen_nocheck . . . 303
bnr_get_mod . . . 303
bnr_get_nf . . . 303
bnr_get_no . . . 303
bnr_sanitize . . . 328
bnr_subgroup_check . . . 328
bnr_subgroup_sanitize . . . 328
bnr_vecchar_sanitize . . . 328
both_odd . . . 82
boundfact . . . 176
BPSW_isprime . . . 181
BPSW_psp . . . 181
brent_kung_optpow . . . 219
brute . . . 272
buchimag . . . 332
Buchray . . . 327
buchreal . . . 332

C

CATCH_ALL . . . 46
cbezout . . . 102
cbrtr . . . 258
cbrtr_abs . . . 259
cb_pari_ask_confirm . . . 55, 56
cb_pari_break_loop . . . 55
cb_pari_display_hist . . . 55
cb_pari_err_handle . . . 55, 279
cb_pari_err_recover . . . 56
cb_pari_handle_exception . . . 55
cb_pari_init_histfile . . . 55
cb_pari_is_interactive . . . 55, 57
cb_pari_long_help . . . 56
cb_pari_pre_recover . . . 55
cb_pari_quit . . . 55
cb_pari_sigint . . . 55
cb_pari_start_output . . . 55
cb_pari_whatnow . . . 56
ceildivuu . . . 99
ceilr . . . 89
ceil_safe . . . 89
centerlift . . . 225
centerlift0 . . . 225
centermod . . . 240
centermodii . . . 97
centermod_i . . . 240
cgcd . . . 101
cgetalloc . . . 67
cgetc . . . 23, 58, 67, 86, 261
cgetg . . . 22, 24, 58, 67
cgetg_block . . . 73
cgetg_copy . . . 58
cgeti . . . 23, 58, 67, 86
cgetineg . . . 86
cgetipos . . . 86
cgetr . . . 23, 58, 67, 86
cgetr_block . . . 73
cgiv . . . 17, 68
character string . . . 32
character . . . 346
characteristic . . . 251
charconj . . . 346
chardiv . . . 347
charmul . . . 346
chartoGENstr . . . 269
char_check . . . 347
char_denormalize . . . 346
char_normalize . . . 346
char_renormalize . . . 346
char_simplify . . . 346
checkabgrp . . . 300
checkalg . . . 349
checkbid . . . 299
checkbid_i . . . 299

checkbnf . . . 299
checkbnf_i . . . 299
checkbnr . . . 299
checkbnr_i . . . 299
checkell . . . 353
checkell5 . . . 353
checkellisog . . . 354
checkellpt . . . 353
checkellpt_i . . . 353
checkell_Fq . . . 354
checkell_i . . . 353
checkell_Q . . . 354
checkell_Qp . . . 354
checkfarey_i . . . 375
checkgal . . . 299
checkgroup . . . 264
checkgroupelts . . . 264
checkhasse . . . 349
checklat . . . 349
checkMF . . . 375
checkMF_i . . . 375
checkmf_i . . . 375
checkmodpr . . . 300
checkms . . . 373
checkmspadic . . . 373
checknf . . . 299
checknfelt_mod . . . 300
checknf_i . . . 299
checkprid . . . 300
checkprid_i . . . 300
checkrnf . . . 299
checkrnf_i . . . 299
checksqmat . . . 299
checkznstar_i . . . 299
check_arith_all . . . 179
check_arith_non0 . . . 179
check_arith_pos . . . 179
check_ecppcert . . . 181
check_modinv . . . 367
check_quaddisc . . . 334
check_quaddisc_imag . . . 334
check_quaddisc_real . . . 334
check_ZKmodule . . . 300
check_ZKmodule_i . . . 300
chinese1 . . . 162
chinese1_coprime_Z . . . 162
chk_gc_upto . . . 72
classno . . . 334
classno2 . . . 334
clcm . . . 101
cleanroots . . . 204, 252
clean_Z_factor . . . 179
clone . . . 72
clone . . . 14, 25
CLONEBIT . . . 65
closemodinvertible . . . 346
closure . . . 76
closure . . . 32
closuretoinl . . . 292
closure_arity . . . 32
closure_callgen0 . . . 289
closure_callgen0prec . . . 289
closure_callgen1 . . . 76, 289
closure_callgen1prec . . . 289
closure_callgen2 . . . 289
closure_callgenall . . . 289
closure_callgenvec . . . 289
closure_callgenvecdef . . . 289
closure_callgenvecdefprec . . . 289
closure_callgenvecprec . . . 289
closure_callvoid1 . . . 289
closure_context . . . 292
closure_deriv . . . 291
closure_derivn . . . 291
closure_disassemble . . . 289
closure_err . . . 292
closure_evalbrk . . . 290
closure_evalgen . . . 76, 289
closure_evalnobrk . . . 289
closure_evalres . . . 290
closure_evalvoid . . . 76, 290
closure_func_err . . . 55
closure_is_variadic . . . 32
cmpii . . . 93
cmpir . . . 93
cmpis . . . 93
cmpiu . . . 93
cmpri . . . 94
cmprr . . . 94
cmprs . . . 94
cmpsi . . . 93
cmpsr . . . 93
cmpss . . . 93
cmpui . . . 93
cmpuu . . . 93
cmp_Flx . . . 239
cmp_nodata . . . 238
cmp_padic . . . 239

cmp_prime_ideal 239
cmp_prime_over_p 239
cmp_RgX 239
cmp_universal 194, 235, 238
colorname_to_rgb 381
colors 271, 272
color_to_rgb 381
coltrunc_init 58
column vector 31
col_ei 228
compile_str 56, 291
complex number 29
compo 63
conjclasses_algcenter 351
conjclasses_repr 265
conjvec 252, 261
conj_i 249
constant_coeff 30, 64
constbern 262
constcatalan 262
consteuler 262
constlog2 262
constpi 262
constzeta 262
const_col 285
const_F2v 120
const_vec 285
const_vecsmall 286
content 241
conversions 25
coprimes_zv 101
copy 24
copybin_unlink 68
copyifstack 72
copy_bin 67
copy_bin_canon 68
core 180
core2 180
core2partial 180
coredisc2u_fact 106
coredisc2_fact 106
corediscs 106
corepartial 180
coreu 106
coreu_fact 106
cornacchia 337
cornacchia2 337
cornacchia2_sqrt 337
creation 22
cvstop2 224
cvtop 224
cvtop2 224
cxcompotor 223
cxEk 261
cxexpm1 250
cxnorm 250
cxredsl2 261
cxredsl2_i 261
cxtofp 223
cxtoreal 223
cx_approx0 250
cx_approx_equal 250
cyc2elts 347
cyclicgroup 264
cyclic_perm 263
cyc_get_expo 300
cyc_normalize 346
cyc_pow 263
cyc_pow_perm 263
c_ERR 271
c_HELP 272
c_HIST 271
C_INCLUDE_PATH 13
c_INPUT 272
c_NONE 272
c_OUTPUT 272
c_PROMPT 272
c_TIME 272

D

datadir 80
dbgGEN 38
dbg_fill_stack 72
dbg_gc_upto 72
dbg_pari_heap 39
dbg_stack 72
dblcabs 250
dblcarg 250
dblclog 250
dblcnorm 250
dblcoro526 371
dblexpo 223
dbllambertW0 250
dbllambertW_1 250
dbllemma526 371
dbllog2 250
dbllog2r 250

dblmantissa 223
dblmodulus 223
dbltor 25, 223
debug 39
debugging 38
DEBUGLEVEL 39
DEBUGMEM 39
debugmem 39
default0 54
DEFAULTPREC 16, 64
deg1pol 228
deg1pol_shallow 233
deg1_from_roots 233
deg2pol_shallow 233
degpol 30, 61, 251
degree 30, 251
delete_var 34, 74
denom_i 241
deplin 345
derivnum0 290
derivser 252
destruction 17
detint 185
diagonal_shallow 233
dicyclicgroup 264
direuler0 290
dirpowerssum0 290
discrayabslist 333
discrayabslistarch 333
disc_best_modinv 367
diviiexact 97
diviiround 89
divisors 40
divisorsu 107
divisorsu_fact 107
divisorsu_fact_factored 107
divisorsu_moebius 107
divisors_fact_degmax 107
divis_rem 99
diviuexact 97
diviuuexact 97
divll 81
divll_pre 84
divmpr 95
divrmp 95
divrunextu 239
divsBIL 65
divsi_rem 99
divss_rem 99
div_content 242
dvdii 98
dvdiiz 98
dvdis 98
dvdisz 98
dvdiu 98
dvdiuz 98
dvdsi 98
dvdui 98
dvmdii 98
dvmdis 98
dvmdsBIL 65
dvmdsi 98
dvmdss 98
dvmduBIL 65
dynamic array 281
d_ACKNOWLEDGE 293, 295, 296
d_INITRC 293, 295
d_RETURN 293, 295, 296
d_SILENT 293

E

echo 57
ecpp 181
ecpp0 181
ecppexport 181
ecppisvalid 181
ec_2divpol_evalx 358
ec_3divpol_evalx 358
ec_bmodel 358
ec_dFdx_evalQ 358
ec_dFdy_evalQ 358
ec_dmFdy_evalQ 358
ec_f_evalx 358
ec_half_deriv_2divpol 358
ec_half_deriv_2divpol_evalx 358
ec_h_evalx 358
ec_phi2 358
effective length 27
ei_multable 310
elff_get_o 357
ell 353
ellanal_globalred 355
ellanal_globalred_all 355
ellap 358
ellap_CM_fast 354
ellbasechar 358
ellchangeinvert 358

elldatagenerators 355
elleulerf 356
ellff_get_a4a6 357
ellff_get_card 357
ellff_get_D 357
ellff_get_field 357
ellff_get_gens 357
ellff_get_group 357
ellff_get_m 357
ellff_get_o 357
ellff_get_p 356
ellheightoo 355
ellinf 358
ellinit 353
ellintegralmodel 355
ellintegralmodel_i 355
ellisogeny 354
ellnf_get_bnf 356
ellnf_get_nf 356
ellnf_vecarea 356
ellnf_veceta 356
ellnf_vecomega 356
ellorder 355
ellorder_Q 355
ellQp_ab 356
ellQp_AGM 356
ellQp_Ei 262
ellQp_get_p 356
ellQp_get_prec 356
ellQp_get_zero 356
ellQp_L 356
ellQp_q 356
ellQp_root 356
ellQp_Tate_uniformization 356
ellQp_u 356
ellQp_u2 356
ellQtwist_bsdperiod 355
ellQ_get_CM 354, 361
ellQ_get_N 354
ellQ_get_Nfa 354
ellrootno_global 355
ellR_ab 357
ellR_area 357
ellR_eta 357
ellR_get_prec 357
ellR_get_sign 358
ellR_omega 357
ellR_roots 357
ellsupersingularj_FpXQ 365
elltors 356
elltors0 356
elltors_psylow 356
elltrace_extension 359
ell_get_a1 354
ell_get_a2 354
ell_get_a3 354
ell_get_a4 354
ell_get_a6 354
ell_get_b2 354
ell_get_b4 354
ell_get_b6 354
ell_get_b8 354
ell_get_c4 354
ell_get_c6 354
ell_get_disc 354
ell_get_j 354
ell_get_type 353, 359
ell_is_inf 358
ell_is_integral 354
eltabstorel 330
eltabstorel_lift 330
eltreltoabs 330
embednorm_T2 325
embed_disc 325
embed_norm 325
embed_roots 325
embed_T2 325
equali1 235
equalii 94
equalim1 236
equalis 94
equaliu 94
equalrr 94
equalsi 94
equalui 94
error context 32
error 37
error 277
err_flush 75
err_get_compo 46
err_get_num 46
err_printf 75
etaquotype 370
eta_product_ZXn 202
eta_ZXn 202
eulerf_gchar 348
eulerphiu 106
eulerphiu_fact 106

evalexpo 62
evallg 61
evallgefint 62
evalprecp 62
evalsigne 61
evaltyp 61
evalvalp 62
evalvalser 62
evalvarn 61
exp1r_abs 259
expi 60
expIPiC 260
expIPiQ 260
expIPiR 260
expIr 260
expIxy 260
expo 28, 31, 60
EXPOBITS 65
EXPOnumBITS 65
expu 96
extract0 284
e_ALARM 57
e_DOMAIN 179
e_MISC 37
e_TYPE 179

F

F2c_to_mod 160
F2c_to_ZC 120
F2Ms_colelim 196
F2Ms_ker 196
F2Ms_to_F2m 196
F2m_clear 119
F2m_coeff 119
F2m_copy 119
F2m_deplin 121
F2m_det 121
F2m_det_sp 121
F2m_F2c_gauss 119
F2m_F2c_invimage 119
F2m_F2c_mul 119
F2m_flip 119
F2m_gauss 119
F2m_image 119
F2m_indexrank 120
F2m_inv 121
F2m_invimage 119
F2m_ker 121
F2m_ker_sp 121
F2m_mul 120
F2m_powu 120
F2m_rank 120
F2m_row 119
F2m_rowslice 119
F2m_set 119
F2m_suppl 120
F2m_to_F2Ms 196
F2m_to_Flm 120
F2m_to_mod 160
F2m_to_ZM 120
F2m_transpose 119
F2v_add_inplace 120
F2v_and_inplace 120
F2v_clear 119
F2v_coeff 119
F2v_copy 119
F2v_dotproduct 121
F2v_ei 120
F2v_equal0 119
F2v_flip 119
F2v_hamming 120
F2v_negimply_inplace 120
F2v_or_inplace 120
F2v_set 119
F2v_slice 119
F2v_subset 120
F2v_to_F2x 154
F2v_to_Flv 120
F2xC_to_FlxC 174
F2xC_to_ZXC 174
F2xn_div 157
F2xn_inv 157
F2xn_red 157
F2xqE_add 363
F2xqE_changepoint 363
F2xqE_changepointinv 363
F2xqE_dbl 363
F2xqE_log 363
F2xqE_mul 363
F2xqE_neg 363
F2xqE_order 363
F2xqE_sub 363
F2xqE_tatepairing 364
F2xqE_weilpairing 364
F2xqM_deplin 157
F2xqM_det 157
F2xqM_F2xqC_gauss 157

F2xqM_F2xqC_invimage 157
F2xqM_F2xqC_mul 157
F2xqM_gauss 157
F2xqM_image 157
F2xqM_indexrank 157
F2xqM_inv 157
F2xqM_invimage 157
F2xqM_ker 157
F2xqM_mul 157
F2xqM_rank 157
F2xqM_suppl 157
F2xqV_roots_to_pol 156
F2xqXQV_red 160
F2xqXQ_autpow 160
F2xqXQ_auttrace 160
F2xqXQ_inv 159
F2xqXQ_invsafe 159
F2xqXQ_mul 160
F2xqXQ_pow 160
F2xqXQ_powers 160
F2xqXQ_sqr 160
F2xqXV_prod 159
F2xqX_ddf 159
F2xqX_degfact 159
F2xqX_disc 159
F2xqX_div 159
F2xqX_divrem 159
F2xqX_extgcd 159
F2xqX_F2xqXQV_eval 159
F2xqX_F2xqXQ_eval 159
F2xqX_F2xq_mul 159
F2xqX_F2xq_mul_to_monic 159
F2xqX_factor 159
F2xqX_factor_squarefree 159
F2xqX_Frobenius 159
F2xqX_gcd 159
F2xqX_get_red 158
F2xqX_halfgcd 159
F2xqX_halfgcd_all 159
F2xqX_invBarrett 159
F2xqX_ispower 159
F2xqX_mul 159
F2xqX_nbroots 159
F2xqX_normalize 159
F2xqX_powu 159
F2xqX_red 158
F2xqX_rem 159
F2xqX_resultant 159
F2xqX_roots 159
F2xqX_sqr 159
F2xq_Artin_Schreier 156
F2xq_autpow 156
F2xq_conjvec 156
F2xq_div 156
F2xq_ellcard 363
F2xq_ellgens 363
F2xq_ellgroup 363
F2xq_elltwist 363
F2xq_inv 156
F2xq_invsafe 156
F2xq_log 156
F2xq_matrix_pow 156
F2xq_mul 156
F2xq_order 156
F2xq_pow 156
F2xq_powers 156
F2xq_powu 156
F2xq_pow_init 156
F2xq_pow_table 156
F2xq_sqr 156
F2xq_sqrt 156
F2xq_sqrtn 156
F2xq_sqrt_fast 156
F2xq_trace 156
F2xV_to_F2m 174
F2xV_to_FlxV_inplace 172
F2xV_to_ZXV_inplace 172
F2xXC_to_ZXXC 158
F2xXV_to_F2xM 157
F2xX_add 157
F2xX_deriv 158
F2xX_F2x_add 158
F2xX_F2x_mul 158
F2xX_renormalize 157
F2xX_shift 157
F2xX_to_F2xC 157
F2xX_to_FlxX 157
F2xX_to_Kronecker 158
F2xX_to_ZXX 157
F2xY_degreex 157
F2xY_F2xqV_evalx 158
F2xY_F2xq_evalx 158
F2x_1_add 155
F2x_add 155
F2x_clear 154
F2x_coeff 154
F2x_copy 154
F2x_ddf 156

F2x_deflate . . . 155
F2x_degfact . . . 155
F2x_degree . . . 155
F2x_deriv . . . 155
F2x_div . . . 155
F2x_divrem . . . 155
F2x_equal . . . 155
F2x_equal1 . . . 155
F2x_eval . . . 155
F2x_even_odd . . . 155
F2x_extgcd . . . 155
F2x_F2xqV_eval . . . 156
F2x_F2xq_eval . . . 156
F2x_factor . . . 156
F2x_factor_squarefree . . . 156
F2x_flip . . . 154
F2x_Frobenius . . . 155
F2x_gcd . . . 155
F2x_get_red . . . 154
F2x_halfgcd . . . 155
F2x_issquare . . . 155
F2x_is_irred . . . 155
F2x_matFrobenius . . . 155
F2x_mul . . . 155
F2x_recip . . . 155
F2x_rem . . . 155
F2x_renormalize . . . 155
F2x_set . . . 154
F2x_shift . . . 155
F2x_sqr . . . 155
F2x_sqrt . . . 155
F2x_Teichmuller . . . 156
F2x_to_F2v . . . 174
F2x_to_F2xX . . . 154
F2x_to_Flx . . . 154
F2x_to_ZX . . . 154
F2x_valrem . . . 155
F3c_to_mod . . . 160
F3c_to_ZC . . . 121
F3m_coeff . . . 121
F3m_copy . . . 121
F3m_ker . . . 121
F3m_ker_sp . . . 121
F3m_mul . . . 121
F3m_rank . . . 121
F3m_row . . . 121
F3m_set . . . 121
F3m_to_Flm . . . 122
F3m_to_mod . . . 160
F3m_to_ZM . . . 122
F3m_transpose . . . 121
F3v_clear . . . 121
F3v_coeff . . . 121
F3v_dotproduct . . . 121
F3v_dotsquare . . . 121
F3v_set . . . 121
F3v_to_Flv . . . 121
factmod . . . 161
factor . . . 353
factorback . . . 245
factoredpolred . . . 333
factoredpolred2 . . . 333
factorial_Fl . . . 83
factorial_Fp . . . 105
factorial_lval . . . 92
factorint . . . 178
factoru . . . 177
factoru_pow . . . 177
factor_Aurifeuille . . . 177
factor_Aurifeuille_prime . . . 176
factor_pn_1 . . . 176
factor_pn_1_limit . . . 176
factor_proven . . . 180
famat . . . 311
famatsmall_reduce . . . 313
famatV_factorback . . . 312
famatV_zv_factorback . . . 312
famat_div . . . 312
famat_div_shallow . . . 312
famat_idealfactor . . . 313
famat_inv . . . 312
famat_inv_shallow . . . 312
famat_makecoprime . . . 327
famat_mul . . . 312
famat_mulpow . . . 312
famat_mulpows_shallow . . . 312
famat_mulpow_shallow . . . 312
famat_mul_shallow . . . 312
famat_nfvalrem . . . 313
famat_pow . . . 312
famat_pows_shallow . . . 312
famat_pow_shallow . . . 312
famat_reduce . . . 311, 313
famat_remove_trivial . . . 313
famat_sqr . . . 312
famat_to_nf . . . 313
famat_to_nf_moddivisor . . . 326
famat_to_nf_modideal_coprime . . . 327

famat_Z_gcd . . . 312
fetch_user_var . . . 33, 74
fetch_var . . . 34, 74
fetch_var_higher . . . 34
fetch_var_value . . . 34, 74
FFM_deplin . . . 257
FFM_det . . . 257
FFM_FFC_gauss . . . 257
FFM_FFC_invimage . . . 257
FFM_FFC_mul . . . 257
FFM_gauss . . . 257
FFM_image . . . 257
FFM_indexrank . . . 257
FFM_inv . . . 257
FFM_invimage . . . 257
FFM_ker . . . 257
FFM_mul . . . 257
FFM_rank . . . 257
FFM_suppl . . . 257
FFV_roots_to_pol . . . 257
FFXQ_inv . . . 258
FFXQ_minpoly . . . 258
FFXQ_mul . . . 258
FFXQ_sqr . . . 258
FFX_add . . . 255
FFX_ddf . . . 256
FFX_degfact . . . 256
FFX_disc . . . 256
FFX_extgcd . . . 256
FFX_factor . . . 256
FFX_factor_squarefree . . . 256
FFX_gcd . . . 256
FFX_halfgcd . . . 256
FFX_halfgcd_all . . . 256
FFX_ispower . . . 256
FFX_mul . . . 255
FFX_nbroots . . . 256
FFX_preimage . . . 257
FFX_preimagerel . . . 257
FFX_rem . . . 256
FFX_resultant . . . 256
FFX_roots . . . 256
FFX_sqr . . . 255
FF_1 . . . 253
FF_add . . . 254
FF_charpoly . . . 255
FF_conjvec . . . 255
FF_div . . . 254
FF_ellcard . . . 359
FF_ellcard_SEA . . . 359
FF_ellgens . . . 359
FF_ellgroup . . . 359
FF_elllog . . . 359
FF_ellmul . . . 359
FF_ellorder . . . 359
FF_ellrandom . . . 359
FF_elltatepairing . . . 359
FF_elltwist . . . 359
FF_ellweilpairing . . . 359
FF_equal . . . 254
FF_equal0 . . . 254
FF_equal1 . . . 254
FF_equalm1 . . . 254
FF_f . . . 253
FF_Frobenius . . . 254
FF_gen . . . 253
FF_inv . . . 254
FF_ispower . . . 255
FF_issquare . . . 255
FF_issquareall . . . 255
FF_log . . . 255
FF_map . . . 255
FF_minpoly . . . 255
FF_mod . . . 253
FF_mul . . . 254
FF_mul2n . . . 254
FF_neg . . . 254
FF_neg_i . . . 254
FF_norm . . . 255
FF_order . . . 255
FF_p . . . 253
ff_parse_Tp . . . 128
FF_pow . . . 254
FF_primroot . . . 255
FF_p_i . . . 253
FF_q . . . 253
FF_Q_add . . . 254
FF_samefield . . . 254
FF_sqr . . . 254
FF_sqrt . . . 255
FF_sqrtn . . . 255
FF_sub . . . 254
FF_to_F2xq . . . 253
FF_to_F2xq_i . . . 253
FF_to_Flxq . . . 253
FF_to_Flxq_i . . . 253
FF_to_FpXQ . . . 253
FF_to_FpXQ_i . . . 253

FF_trace . . . 255
FF_var . . . 253
FF_zero . . . 254
FF_Z_add . . . 254
FF_Z_mul . . . 254
FF_Z_Z_muldiv . . . 255
file_is_binary . . . 272
finite field element . . . 29
fixlg . . . 71, 87
Flc_Flv_mul . . . 116
Flc_FpV_mul . . . 117
Flc_lincomb1_inplace . . . 117
Flc_to_mod . . . 160
Flc_to_ZC . . . 172
Flc_to_ZC_inplace . . . 172
Fle_add . . . 361
Fle_changepoint . . . 362
Fle_changepointinv . . . 362
Fle_dbl . . . 361
Fle_log . . . 362
Fle_mul . . . 362
Fle_mulu . . . 362
Fle_order . . . 362
Fle_sub . . . 361
Fle_tatepairing . . . 362
Fle_to_Flj . . . 362
Fle_weilpairing . . . 362
FljV_factorback_pre . . . 363
Flj_add_pre . . . 362
Flj_changepointinv_pre . . . 362
Flj_dbl_pre . . . 362
Flj_mulu_pre . . . 362
Flj_neg . . . 362
Flj_to_Fle . . . 362
Flj_to_Fle_pre . . . 362
Flm_add . . . 118
Flm_adjoint . . . 118
Flm_center . . . 116
Flm_charpoly . . . 118
Flm_copy . . . 116
Flm_deplin . . . 118
Flm_det . . . 118
Flm_det_sp . . . 118
Flm_Flc_gauss . . . 118
Flm_Flc_invimage . . . 118
Flm_Flc_mul . . . 116
Flm_Flc_mul_pre . . . 116
Flm_Flc_mul_pre_Flx . . . 116
Flm_Fl_add . . . 116
Flm_Fl_mul . . . 117
Flm_Fl_mul_inplace . . . 117
Flm_Fl_mul_pre . . . 117
Flm_Fl_sub . . . 116
Flm_gauss . . . 118
Flm_hess . . . 119
Flm_hess_pre . . . 119
Flm_image . . . 119
Flm_indexrank . . . 118
Flm_intersect . . . 119
Flm_intersect_i . . . 119
Flm_inv . . . 118
Flm_invimage . . . 118
Flm_ker . . . 118
Flm_ker_sp . . . 118
Flm_mul . . . 118
Flm_mul_pre . . . 118
Flm_neg . . . 117
Flm_powers . . . 118
Flm_powu . . . 118
Flm_rank . . . 118
Flm_row . . . 118
Flm_sqr . . . 118
Flm_sub . . . 118
Flm_suppl . . . 118
Flm_to_F2m . . . 120
Flm_to_F3m . . . 122
Flm_to_FlxV . . . 173
Flm_to_FlxX . . . 174
Flm_to_mod . . . 160
Flm_to_ZM . . . 173
Flm_to_ZM_inplace . . . 173
Flm_trace . . . 118
Flm_transpose . . . 119
floorr . . . 89
floor_safe . . . 89
flush . . . 271
Flv_add . . . 117
Flv_add_inplace . . . 117, 320
Flv_center . . . 116
Flv_copy . . . 116
Flv_dotproduct . . . 117
Flv_dotproduct_pre . . . 117
Flv_factorback . . . 117
Flv_Flm_polint . . . 144
Flv_Fl_div . . . 117
Flv_Fl_div_inplace . . . 117
Flv_Fl_mul . . . 117
Flv_Fl_mul_inplace . . . 117

Flv_Fl_mul_part_inplace 117
Flv_inv 117
Flv_invVandermonde 144
Flv_inv_inplace 117
Flv_inv_pre 117
Flv_inv_pre_inplace 117
Flv_neg 117
Flv_neg_inplace 117
Flv_polint 144
Flv_prod 117
Flv_prod_pre 117
Flv_roots_to_pol 144
Flv_sub 117
Flv_sub_inplace 117
Flv_sum 117
Flv_to_F2v 120
Flv_to_F3v 122
Flv_to_Flx 173
Flv_to_ZV 173
FlxC_eval_powers_pre 145
FlxC_FlxqV_eval 147
FlxC_FlxqV_eval_pre 147
FlxC_Flxq_eval 147
FlxC_Flxq_eval_pre 147
FlxC_Fl_translate 145
FlxC_neg 145
FlxC_sub 145
FlxC_to_F2xC 174
FlxC_to_ZXC 172
FlxM_eval_powers_pre 145
FlxM_Flx_add_shallow 122
FlxM_Fl_translate 145
FlxM_mul 145
FlxM_neg 145
FlxM_sub 145
FlxM_to_FlxXV 174
FlxM_to_ZXM 172
Flxn_div 145
Flxn_div_pre 145
Flxn_exp 145
Flxn_expint 146
Flxn_inv 145
Flxn_mul 145
Flxn_mul_pre 145
Flxn_red 145
Flxn_sqr 145
Flxn_sqr_pre 145
FlxqC_Flxq_mul 122
FlxqC_red_pre 122
FlxqE_add 364
FlxqE_changepoint 364
FlxqE_changepointinv 364
FlxqE_dbl 364
FlxqE_log 365
FlxqE_mul 364
FlxqE_neg 364
FlxqE_order 364
FlxqE_sub 364
FlxqE_tatepairing 365
FlxqE_weilpairing 365
FlxqE_weilpairing_pre 365
FlxqM_deplin 122
FlxqM_det 122
FlxqM_FlxqC_gauss 122
FlxqM_FlxqC_invimage 122
FlxqM_FlxqC_mul 122
FlxqM_Flxq_mul 122
FlxqM_gauss 122
FlxqM_image 122
FlxqM_indexrank 122
FlxqM_inv 122
FlxqM_invimage 122
FlxqM_ker 122
FlxqM_mul 122
FlxqM_rank 122
FlxqM_red_pre 122
FlxqM_suppl 122
FlxqV_dotproduct 122
FlxqV_dotproduct_pre 122
FlxqV_factorback 146
FlxqV_roots_to_pol 147
FlxqXC_FlxqXQV_eval 152
FlxqXC_FlxqXQV_eval_pre 152
FlxqXC_FlxqXQ_eval 152
FlxqXC_FlxqXQ_eval_pre 152
FlxqXn_div 154
FlxqXn_div_pre 154
FlxqXn_expint 154
FlxqXn_expint_pre 154
FlxqXn_inv 154
FlxqXn_inv_pre 154
FlxqXn_mul 153
FlxqXn_mul_pre 153
FlxqXn_sqr 153
FlxqXn_sqr_pre 154
FlxqXQ_autpow 153
FlxqXQ_autpow_pre 153
FlxqXQ_autsum 153

FlxqXQ_autsum_pre 153
FlxqXQ_auttrace 153
FlxqXQ_auttrace_pre 153
FlxqXQ_div 153
FlxqXQ_div_pre 153
FlxqXQ_halfFrobenius 153
FlxqXQ_inv 153, 159
FlxqXQ_invsafe 153, 159
FlxqXQ_invsafe_pre 153
FlxqXQ_inv_pre 153
FlxqXQ_matrix_pow 153
FlxqXQ_minpoly 153
FlxqXQ_minpoly_pre 153
FlxqXQ_mul 153
FlxqXQ_mul_pre 153
FlxqXQ_pow 153
FlxqXQ_powers 153
FlxqXQ_powers_pre 153
FlxqXQ_powu 153
FlxqXQ_powu_pre 153
FlxqXQ_pow_pre 153
FlxqXQ_sqr 153
FlxqXQ_sqr_pre 153
FlxqXV_prod 151
FlxqX_composedsum 151
FlxqX_ddf 152
FlxqX_ddf_degree 152
FlxqX_degfact 152
FlxqX_disc 151
FlxqX_div 151
FlxqX_divrem 150
FlxqX_divrem_pre 151
FlxqX_div_by_X_x 151
FlxqX_div_by_X_x_pre 151
FlxqX_div_pre 151
FlxqX_dotproduct 151
FlxqX_eval 151
FlxqX_extgcd 151
FlxqX_extgcd_pre 151
FlxqX_factor 152
FlxqX_factor_squarefree 152
FlxqX_factor_squarefree_pre 152
FlxqX_FlxqXQV_eval 152
FlxqX_FlxqXQV_eval_pre 152
FlxqX_FlxqXQ_eval 152
FlxqX_FlxqXQ_eval_pre 152
FlxqX_Flxq_mul 150
FlxqX_Flxq_mul_pre 150
FlxqX_Flxq_mul_to_monic 150
FlxqX_Flxq_mul_to_monic_pre 150
FlxqX_Frobenius 152
FlxqX_Frobenius_pre 152
FlxqX_fromNewton 152
FlxqX_fromNewton_pre 152
FlxqX_gcd 151
FlxqX_gcd_pre 151
FlxqX_get_red 150
FlxqX_get_red_pre 150
FlxqX_halfgcd 151
FlxqX_halfgcd_all 151
FlxqX_halfgcd_all_pre 151
FlxqX_halfgcd_pre 151
FlxqX_invBarrett 151
FlxqX_invBarrett_pre 151
FlxqX_ispower 152
FlxqX_is_squarefree 152
FlxqX_mul 150
FlxqX_mul_pre 150
FlxqX_nbfact 152
FlxqX_nbfact_by_degree 152
FlxqX_nbfact_Frobenius 152
FlxqX_nbroots 152
FlxqX_Newton 152
FlxqX_Newton_pre 152
FlxqX_normalize 150
FlxqX_normalize_pre 150
FlxqX_powu 150
FlxqX_powu_pre 150
FlxqX_red 150
FlxqX_red_pre 150
FlxqX_rem 151
FlxqX_rem_pre 151
FlxqX_resultant 151
FlxqX_resultant_pre 151
FlxqX_roots 152
FlxqX_safegcd 151
FlxqX_saferesultant 151
FlxqX_sqr 150
FlxqX_sqr_pre 150
Flxq_add 146
Flxq_autpow 147
Flxq_autpowers 147
Flxq_autpow_pre 147
Flxq_autsum 147
Flxq_auttrace 147
Flxq_auttrace_pre 147
Flxq_charpoly 148
Flxq_conjvec 148

Flxq_div 146
Flxq_div_pre 146
Flxq_ellcard 364
Flxq_elldivpolmod 365
Flxq_ellgens 364
Flxq_ellgroup 364
Flxq_ellj 364
Flxq_ellj_to_a4a6 364
Flxq_elltwist 364
Flxq_ffisom_inv 147
Flxq_inv 146
Flxq_invsafe 146
Flxq_invsafe_pre 146
Flxq_inv_pre 146
Flxq_is2npower 147
Flxq_issquare 147
Flxq_log 147
Flxq_lroot 147
Flxq_lroot_fast 147
Flxq_lroot_fast_pre 148
Flxq_lroot_pre 147
Flxq_matrix_pow 147
Flxq_matrix_pow_pre 147
Flxq_minpoly 148
Flxq_minpoly_pre 148
Flxq_mul 146
Flxq_mul_pre 146
Flxq_norm 148
Flxq_order 147
Flxq_pow 146
Flxq_powers 146
Flxq_powers_pre 147
Flxq_powu 146
Flxq_powu_pre 146
Flxq_pow_init 146
Flxq_pow_init_pre 146
Flxq_pow_pre 146
Flxq_pow_table 146
Flxq_pow_table_pre 146
Flxq_sqr 146
Flxq_sqrt 147
Flxq_sqrtn 147
Flxq_sqrt_pre 147
Flxq_sqr_pre 146
Flxq_sub 146
Flxq_trace 148
FlxT_red 145
FlxV_composedsum 145
FlxV_factorback 145
FlxV_Flc_mul 144
FlxV_Flv_multieval 145
FlxV_Flx_fromdigits 141
FlxV_prod 145
FlxV_red 145
FlxV_to_Flm 174
FlxV_to_FlxX 174
FlxV_to_ZXV 172
FlxV_to_ZXV_inplace 172
FlxXC_sub 149
FlxXC_to_F2xXC 158
FlxXC_to_ZXXC 172
FlxXM_to_ZXXM 172
FlxXn_red 153
FlxXV_to_FlxM 174
FlxX_add 148
FlxX_blocks 149
FlxX_deriv 148
FlxX_double 148
FlxX_Fl_mul 148
FlxX_invLaplace 148
FlxX_Laplace 148
FlxX_neg 148
FlxX_renormalize 149
FlxX_resultant 149
FlxX_resultant_pre 149
FlxX_shift 149
FlxX_sub 148
FlxX_swap 149
FlxX_to_F2xX 157
FlxX_to_Flm 174
FlxX_to_Flx 174
FlxX_to_FlxC 174
FlxX_to_ZXX 172
FlxX_translate1 149
FlxX_triple 148
FlxYqq_pow 149
FlxY_degreex 148
FlxY_evalx 148
FlxY_evalx_powers_pre 149
FlxY_evalx_pre 149
FlxY_eval_powers_pre 149
FlxY_FlxqV_evalx 149
FlxY_FlxqV_evalx_pre 149
FlxY_Flxq_evalx 149
FlxY_Flxq_evalx_pre 149
FlxY_Flx_add 148
FlxY_Flx_div 148
FlxY_Flx_div_pre 148

FlxY_Flx_mul 148
FlxY_Flx_mul_pre 148
FlxY_Flx_sub 148
FlxY_Flx_translate 149
Flx_add 140
Flx_blocks 143
Flx_composedprod 143
Flx_composedsum 143
Flx_constant 140
Flx_convol 140
Flx_copy 140
Flx_ddf 142
Flx_ddf_pre 142
Flx_deflate 143
Flx_degfact 142, 144
Flx_deriv 140
Flx_diff1 141
Flx_digits 141
Flx_div 140
Flx_divrem 140
Flx_divrem_pre 140
Flx_div_by_X_x 143
Flx_div_pre 140
Flx_dotproduct 143
Flx_dotproduct_pre 143
Flx_double 140
Flx_equal 140
Flx_equal1 140
Flx_eval 143
Flx_eval_powers_pre 143
Flx_eval_pre 143
Flx_extgcd 141
Flx_extgcd_pre 141
Flx_extresultant 143
Flx_extresultant_pre 143
Flx_factcyclo 142
Flx_factor 142
Flx_factorff_irred 142
Flx_factor_squarefree 142
Flx_factor_squarefree_pre 142
Flx_ffintersect 144
Flx_ffisom 142
Flx_Flv_multieval 143
Flx_FlxqV_eval 147
Flx_FlxqV_eval_pre 147
Flx_Flxq_eval 147
Flx_Flxq_eval_pre 147
Flx_FlxY_resultant 149
Flx_Fl_add 140
Flx_Fl_mul 140
Flx_Fl_mul_pre 140
Flx_Fl_mul_to_monic 140
Flx_Fl_sub 140
Flx_Fl_translate 141
Flx_Frobenius 141
Flx_Frobenius_pre 141
Flx_fromNewton 144
Flx_gcd 141
Flx_gcd_pre 141
Flx_get_red 139
Flx_get_red_pre 139
Flx_halfgcd 141
Flx_halfgcd_all 141
Flx_halfgcd_all_pre 141
Flx_halfgcd_pre 141
Flx_halve 140
Flx_inflate 143
Flx_integ 141
Flx_invBarrett 143
Flx_invLaplace 144
Flx_ispower 142
Flx_is_irred 144
Flx_is_smooth 144
Flx_is_smooth_pre 144
Flx_is_squarefree 143
Flx_is_totally_split 144
Flx_Laplace 144
Flx_lead 139
Flx_matFrobenius 141
Flx_matFrobenius_pre 141
Flx_mod_Xn1 142
Flx_mod_Xnm1 142
Flx_mul 140
Flx_mulu 140
Flx_mul_pre 140
Flx_nbfact 144
Flx_nbfact_by_degree 144
Flx_nbfact_Frobenius 144
Flx_nbfact_Frobenius_pre 144
Flx_nbfact_pre 144
Flx_nbroots 144
Flx_neg 140
Flx_neg_inplace 140
Flx_Newton 144
Flx_normalize 142
Flx_oneroot 141
Flx_oneroot_pre 141
Flx_oneroot_split 142

Flx_oneroot_split_pre 142
Flx_powu 140
Flx_powu_pre 140
Flx_recip 143
Flx_red 140
Flx_rem 140
Flx_rem_pre 140
Flx_renormalize 143
Flx_rescale 142
Flx_resultant 143
Flx_resultant_pre 143
Flx_roots 141
Flx_rootsff 142
Flx_roots_pre 141
Flx_shift 143
Flx_splitting 143
Flx_sqr 140
Flx_sqr_pre 140
Flx_sub 140
Flx_Teichmuller 144
Flx_to_F2x 154
Flx_to_Flv 173
Flx_to_FlxX 172
Flx_to_ZX 172
Flx_to_ZX_inplace 172
Flx_translate1 141
Flx_translate1_basecase 141
Flx_triple 140
Flx_unscale 142
Flx_val 143
Flx_valrem 143
Fly_to_FlxY 174
Fl_2gener_pre 85
Fl_2gener_pre_i 85
Fl_add 82
Fl_addmulmul_pre 84
Fl_addmul_pre 84
Fl_center 82
Fl_div 83
Fl_double 82
Fl_elldisc 361
Fl_elldisc_pre 361
Fl_ellj 361
Fl_ellj_pre 361
Fl_ellj_to_a4a6 361
Fl_ellptors 361
Fl_elltrace 361
Fl_elltrace_CM 361
Fl_elltwist 361
Fl_elltwist_disc 361
Fl_halve 82
Fl_inv 82
Fl_invgen 83
Fl_invsafe 82
Fl_log 83
Fl_log_pre 84
Fl_mul 82
Fl_mul_pre 84
Fl_neg 82
Fl_order 83
Fl_powers 83
Fl_powers_pre 84
Fl_powu 83
Fl_powu_pre 84
Fl_sqr 82
Fl_sqrt 83
Fl_sqrtl 83
Fl_sqrtl_pre 84
Fl_sqrtn 83
Fl_sqrtn_pre 84
Fl_sqrt_pre 84
Fl_sqrt_pre_i 85
Fl_sqr_pre 84
Fl_sub 82
Fl_to_Flx 173
Fl_triple 82
forallsubset_init 42
forcomposite 41
forcomposite_init 41
forcomposite_next 41
fordiv 40, 290
fordivfactored 290
foreachpari 290
forell 40
forell(ell,a,b,,flag) 40
forell0 290
forfactored 290
forksubset_init 42
format 38
forpari 290
forpart 41
forpart0 290
forpart_init 41
forpart_next 41
forpart_prev 41
forpart_t 41
forperm 41
forperm0 290

forperm_init 41
forperm_next 41
forprime 41
forprimestep 41
forprimestep_init 42, 182
forprime_init 42, 43, 181, 182
forprime_next 42, 182
forprime_t 42, 43
forqfvec 40
forqfvec1 40
forsquarefree 290
forstep 290
forsubgroup 40
forsubgroup(H = G, B,) 40
forsubgroup0 290
forsubset 42
forsubset0 290
forsubset_init 42
forsubset_next 42
forvec 41
forvec_init 41
forvec_next 41
FpC_add 114
FpC_center 113
FpC_center_inplace 114
FpC_FpV_mul 114
FpC_Fp_mul 114
FpC_ratlift 165
FpC_red 113
FpC_sub 114
FpC_to_mod 160
FpE_add 360
FpE_changepoint 360
FpE_changepointinv 360
FpE_dbl 360
FpE_log 361
FpE_mul 360
FpE_neg 360
FpE_order 361
FpE_sub 360
FpE_tatepairing 361
FpE_to_FpJ 362
FpE_to_mod 361
FpE_weilpairing 361
FpJ_add 362
FpJ_dbl 362
FpJ_mul 362
FpJ_neg 362
FpJ_to_FpE 362
FpMs_FpCs_solve 197
FpMs_FpCs_solve_safe 197
FpMs_FpC_mul 196
FpMs_leftkernel_elt 196
FpM_add 114
FpM_center 113
FpM_center_inplace 114
FpM_charpoly 115
FpM_deplin 115
FpM_det 115
FpM_FpC_gauss 115
FpM_FpC_invimage 115
FpM_FpC_mul 114
FpM_FpC_mul_FpX 114
FpM_Fp_mul 114
FpM_gauss 115
FpM_hess 115
FpM_image 115
FpM_indexrank 115
FpM_intersect 115
FpM_intersect_i 115
FpM_inv 115
FpM_invimage 115
FpM_ker 115
FpM_mul 114
FpM_powu 114
FpM_rank 115
FpM_ratlift 165
FpM_red 113
FpM_sub 114
FpM_suppl 115
FpM_to_mod 160
FpM_trace 115
FpVV_to_mod 160
FpV_add 114
FpV_dotproduct 114
FpV_dotsquare 114
FpV_factorback 114
FpV_FpC_mul 114
FpV_FpMs_mul 196
FpV_FpM_polint 125
FpV_inv 103
FpV_invVandermonde 125
FpV_polint 125
FpV_prod 103
FpV_red 113
FpV_roots_to_pol 125
FpV_sub 114
FpV_to_mod 160

FpXC_center . . . 132
FpXC_FpXQV_eval . . . 131
FpXC_FpXQ_eval . . . 131
FpXC_to_mod . . . 160
FpXk_gcd . . . 133
FpXk_red . . . 133
FpXM_center . . . 132
FpXM_FpXQV_eval . . . 131
FpXM_to_mod . . . 160
FpXn_div . . . 131
FpXn_exp . . . 131
FpXn_expint . . . 132
FpXn_inv . . . 131
FpXn_mul . . . 131
FpXn_sqr . . . 131
FpXQC_to_mod . . . 160
FpXQE_add . . . 366
FpXQE_changepoint . . . 366
FpXQE_changepointinv . . . 366
FpXQE_dbl . . . 366
FpXQE_log . . . 366
FpXQE_mul . . . 366
FpXQE_neg . . . 366
FpXQE_order . . . 366
FpXQE_sub . . . 366
FpXQE_tatepairing . . . 366
FpXQE_weilpairing . . . 366
FpXQM_autsum . . . 131
FpXQV_ellcharpoly . . . 365
FpXQXn_div . . . 135
FpXQXn_exp . . . 135
FpXQXn_expint . . . 136
FpXQXn_inv . . . 135
FpXQXn_mul . . . 135
FpXQXn_sqr . . . 135
FpXQXQ_autpow . . . 137
FpXQXQ_autsum . . . 137
FpXQXQ_auttrace . . . 137
FpXQXQ_div . . . 136
FpXQXQ_halfFrobenius . . . 136
FpXQXQ_inv . . . 136
FpXQXQ_invsafe . . . 136
FpXQXQ_matrix_pow . . . 137
FpXQXQ_minpoly . . . 137
FpXQXQ_mul . . . 136
FpXQXQ_pow . . . 136
FpXQXQ_powers . . . 136
FpXQXQ_sqr . . . 136
FpXQXT_red . . . 134
FpXQXV_FpXQX_fromdigits . . . 135
FpXQXV_prod . . . 135
FpXQXV_red . . . 134
FpXQX_ddf . . . 138
FpXQX_ddf_degree . . . 138
FpXQX_degfact . . . 138
FpXQX_digits . . . 135
FpXQX_disc . . . 135
FpXQX_div . . . 135
FpXQX_divrem . . . 135
FpXQX_div_by_X_x . . . 135
FpXQX_dotproduct . . . 135
FpXQX_extgcd . . . 135
FpXQX_factor . . . 138
FpXQX_factor_squarefree . . . 138
FpXQX_FpXQXQV_eval . . . 136
FpXQX_FpXQXQ_eval . . . 135
FpXQX_FpXQ_mul . . . 135
FpXQX_FpXQ_translate . . . 135
FpXQX_Frobenius . . . 139
FpXQX_gcd . . . 135
FpXQX_get_red . . . 136
FpXQX_halfgcd . . . 135
FpXQX_halfgcd_all . . . 135
FpXQX_invBarrett . . . 135
FpXQX_ispower . . . 138
FpXQX_is_squarefree . . . 137
FpXQX_mul . . . 134
FpXQX_nbfact . . . 139
FpXQX_nbfact_Frobenius . . . 139
FpXQX_nbroots . . . 139
FpXQX_normalize . . . 133
FpXQX_powu . . . 135
FpXQX_red . . . 134
FpXQX_rem . . . 135
FpXQX_renormalize . . . 134
FpXQX_resultant . . . 135
FpXQX_roots . . . 137
FpXQX_roots_mult . . . 137
FpXQX_split_part . . . 138
FpXQX_sqr . . . 135
FpXQX_to_mod . . . 160
FpXQ_add . . . 129
FpXQ_autpow . . . 131
FpXQ_autpowers . . . 131
FpXQ_autsum . . . 131
FpXQ_auttrace . . . 131
FpXQ_charpoly . . . 130
FpXQ_conjvec . . . 130

FpXQ_div . . . 129
FpXQ_ellcard . . . 365
FpXQ_ellcard_supersingular . . . 365
FpXQ_elldivpol . . . 366
FpXQ_ellgens . . . 366
FpXQ_ellgroup . . . 365
FpXQ_ellj . . . 365
FpXQ_elljissupersingular . . . 365
FpXQ_elltwist . . . 366
FpXQ_ffisom_inv . . . 139
FpXQ_inv . . . 129
FpXQ_invsafe . . . 129
FpXQ_issquare . . . 129
FpXQ_log . . . 129, 130, 147
FpXQ_matrix_pow . . . 131
FpXQ_minpoly . . . 130
FpXQ_mul . . . 129
FpXQ_norm . . . 130
FpXQ_order . . . 129, 147
FpXQ_pow . . . 129
FpXQ_powers . . . 131
FpXQ_powu . . . 129
FpXQ_red . . . 129
FpXQ_sqr . . . 129
FpXQ_sqrt . . . 129
FpXQ_sqrtn . . . 129, 130, 147
FpXQ_sub . . . 129
FpXQ_trace . . . 130
FpXT_red . . . 123
FpXV_chinese . . . 125
FpXV_composedsum . . . 125
FpXV_factorback . . . 125
FpXV_FpC_mul . . . 125
FpXV_FpX_fromdigits . . . 124
FpXV_prod . . . 125
FpXV_red . . . 123
FpXX_add . . . 132
FpXX_deriv . . . 132
FpXX_FpX_mul . . . 132
FpXX_Fp_mul . . . 132
FpXX_Fp_sub . . . 132
FpXX_halve . . . 132
FpXX_integ . . . 132
FpXX_mulu . . . 132
FpXX_neg . . . 132
FpXX_red . . . 132
FpXX_renormalize . . . 132
FpXX_sub . . . 132
FpXYQQ_pow . . . 133
FpXY_eval . . . 132
FpXY_evalx . . . 132
FpXY_evaly . . . 132
FpXY_FpXQV_evalx . . . 133
FpXY_FpXQ_evalx . . . 133
FpXY_FpXQ_evaly . . . 132
FpXY_Fq_evaly . . . 133
FpX_add . . . 123
FpX_center . . . 124
FpX_center_i . . . 124
FpX_chinese_coprime . . . 125
FpX_composedprod . . . 127
FpX_composedsum . . . 127
FpX_convol . . . 123
FpX_ddf . . . 126
FpX_ddf_degree . . . 126
FpX_degfact . . . 126, 142, 155
FpX_deriv . . . 123
FpX_digits . . . 123
FpX_disc . . . 127
FpX_div . . . 123
FpX_divrem . . . 123
FpX_divu . . . 124
FpX_div_by_X_x . . . 123
FpX_dotproduct . . . 125
FpX_eval . . . 125
FpX_extgcd . . . 124
FpX_extresultant . . . 127
FpX_factcyclo . . . 126
FpX_factor . . . 126
FpX_factorff . . . 138
FpX_factorff_irred . . . 138, 142
FpX_factor_squarefree . . . 126
FpX_ffintersect . . . 138
FpX_ffisom . . . 138, 142
FpX_FpC_nfpoleval . . . 308
FpX_FpV_multieval . . . 125
FpX_FpXQV_eval . . . 131
FpX_FpXQ_eval . . . 131
FpX_FpXV_multirem . . . 125
FpX_FpXY_resultant . . . 127
FpX_Fp_add . . . 124
FpX_Fp_add_shallow . . . 124
FpX_Fp_div . . . 124
FpX_Fp_mul . . . 124
FpX_Fp_mulspec . . . 124
FpX_Fp_mul_to_monic . . . 124
FpX_Fp_sub . . . 124
FpX_Fp_sub_shallow . . . 124

FpX_Fp_translate . . . 124
FpX_Frobenius . . . 124
FpX_fromNewton . . . 127
FpX_gcd . . . 124
FpX_gcd_check . . . 323
FpX_get_red . . . 127
FpX_halfgcd . . . 124
FpX_halfgcd_all . . . 124
FpX_halve . . . 123
FpX_integ . . . 123
FpX_invBarret . . . 127
FpX_invBarrett . . . 125
FpX_invLaplace . . . 127
FpX_ispower . . . 126
FpX_is_irred . . . 126, 155
FpX_is_squarefree . . . 126
FpX_is_totally_split . . . 126
FpX_Laplace . . . 127
FpX_matFrobenius . . . 124
FpX_mul . . . 123
FpX_mulspec . . . 123
FpX_mulu . . . 124
FpX_nbfact . . . 126
FpX_nbfact_Frobenius . . . 126
FpX_nbroots . . . 126
FpX_neg . . . 123
FpX_Newton . . . 127
FpX_normalize . . . 125
FpX_oneroot . . . 126
FpX_oneroot_split . . . 126
FpX_powu . . . 123
FpX_ratlift . . . 165
FpX_red . . . 123
FpX_rem . . . 123
FpX_renormalize . . . 123
FpX_rescale . . . 125
FpX_resultant . . . 127
FpX_roots . . . 126
FpX_rootsff . . . 138, 142
FpX_roots_mult . . . 126
FpX_split_part . . . 126
FpX_sqr . . . 123
FpX_sub . . . 123
FpX_to_mod . . . 160
FpX_valrem . . . 123
Fp_2gener . . . 104
Fp_2gener_i . . . 104
Fp_add . . . 15, 102
Fp_addmul . . . 103
Fp_center . . . 103
Fp_center_i . . . 103
Fp_div . . . 104
Fp_divu . . . 104
Fp_double . . . 103
Fp_ellcard . . . 360
Fp_ellcard_SEA . . . 360
Fp_elldivpol . . . 360
Fp_ellgens . . . 360
Fp_ellgroup . . . 360
Fp_ellj . . . 359
Fp_elljissupersingular . . . 360
Fp_ellj_to_a4a6 . . . 359
Fp_elltwist . . . 360
Fp_factored_order . . . 104
Fp_ffellcard . . . 360
Fp_FpXQ_log . . . 129
Fp_FpX_sub . . . 124
Fp_halve . . . 103
Fp_inv . . . 103
Fp_invgen . . . 103
Fp_invsafe . . . 103
Fp_ispower . . . 104
Fp_issquare . . . 104
Fp_log . . . 104, 130
Fp_modinv_to_j . . . 367
Fp_mul . . . 103
Fp_muls . . . 103
Fp_mulu . . . 103
Fp_neg . . . 102
Fp_order . . . 104
Fp_polmodular_evalx . . . 367
Fp_pow . . . 103
Fp_powers . . . 103
Fp_pows . . . 103
Fp_powu . . . 103
Fp_pow_init . . . 103
Fp_pow_table . . . 103
Fp_ratlift . . . 165
Fp_red . . . 102
Fp_sqr . . . 103
Fp_sqrt . . . 104
Fp_sqrtn . . . 104
Fp_sqrt_i . . . 104
Fp_sub . . . 102
Fp_to_mod . . . 160
FqC_add . . . 115
FqC_FqV_mul . . . 115
FqC_Fq_mul . . . 115

FqC_red 128
FqC_sub 115
FqC_to_FFC 253
FqC_to_FlxqC 115
FqC_to_mod 160
FqM_add 115
FqM_deplin 116
FqM_det 116
FqM_FqC_gauss 116
FqM_FqC_invimage 116
FqM_FqC_mul 116
FqM_gauss 116
FqM_image 116
FqM_indexrank 116
FqM_inv 116
FqM_invimage 116
FqM_ker 116
FqM_mul 116
FqM_rank 116
FqM_red 129
FqM_sub 116
FqM_suppl 116
FqM_to_FFM 253
FqM_to_FlxqM 115
FqM_to_mod 160
FqM_to_nfM 319
FqV_factorback 130
FqV_inv 130
FqV_red 129
FqV_roots_to_pol 137
FqV_to_FFV 253
FqV_to_nfV 319
FqXC_to_mod 161
FqXM_to_mod 161
FqXn_exp 135
FqXn_expint 136
FqXn_inv 135
FqXn_mul 135
FqXn_sqr 135
FqXQ_add 137
FqXQ_div 137
FqXQ_inv 137
FqXQ_invsafe 137
FqXQ_matrix_pow 137
FqXQ_mul 137
FqXQ_pow 137
FqXQ_powers 137
FqXQ_sqr 137
FqXQ_sub 137
FqXY_eval 134
FqXY_evalx 134
FqX_add 133
FqX_ddf 138
FqX_degfact 138
FqX_deriv 134
FqX_div 134
FqX_divrem 134
FqX_div_by_X_x 134
FqX_eval 134
FqX_extgcd 134
FqX_factor 138
FqX_factor_squarefree 138
FqX_Fp_mul 133
FqX_Fq_add 133
FqX_Fq_mul 133
FqX_Fq_mul_to_monic 133
FqX_Fq_sub 133
FqX_Fq_translate 134
FqX_gcd 134
FqX_get_red 136
FqX_halfgcd 134
FqX_halve 133
FqX_integ 134
FqX_ispower 138
FqX_is_squarefree 137
FqX_mul 133
FqX_mulu 133
FqX_nbfact 139
FqX_nbroots 139
FqX_neg 133
FqX_normalize 133
FqX_powu 134
FqX_red 128
FqX_rem 134
FqX_roots 138
FqX_sqr 133
FqX_sub 133
FqX_to_FFX 253
FqX_to_mod 161
FqX_to_nfX 319
Fq_add 129
Fq_div 130
Fq_ellcard_SEA 365
Fq_ellcard_supersingular 365
Fq_elldivpolmod 366
Fq_elljissupersingular 365
Fq_Fp_mul 130
Fq_halve 130

Fq_inv . . . 130
Fq_invsafe . . . 130
Fq_ispower . . . 130
Fq_issquare . . . 130
Fq_log . . . 130
Fq_mul . . . 130
Fq_mulu . . . 130
Fq_neg . . . 130
Fq_neg_inv . . . 130
Fq_pow . . . 130
Fq_powu . . . 130
Fq_red . . . 128
Fq_sqr . . . 130
Fq_sqrt . . . 130
Fq_sqrtn . . . 130
Fq_sub . . . 129
Fq_to_FF . . . 253
Fq_to_FpXQ . . . 128
Fq_to_nf . . . 318
fractor . . . 223
fractor_lg . . . 223
Frobeniusform . . . 194
fromdigitsu . . . 90
fromdigits_2k . . . 91
fujiwara_bound . . . 252
fujiwara_bound_real . . . 252
fun(E, ell) . . . 40
fun(E, H) . . . 40
functions_basic . . . 54
functions_default . . . 54
functions_gp . . . 54
fuse_Z_factor . . . 179
f_PRETTYMAT . . . 269
f_RAW . . . 269, 270
f_TEX . . . 269, 270

G

gabs . . . 242
gadd . . . 243
gaddgs . . . 15, 243
gaddsg . . . 15, 243
galoisexport . . . 267
galoisidentify . . . 267
galoisinit . . . 264, 333
galois_group . . . 264
gal_get_den . . . 333
gal_get_e . . . 333
gal_get_gen . . . 333
gal_get_group . . . 333
gal_get_invvdm . . . 333
gal_get_mod . . . 333
gal_get_orders . . . 333
gal_get_p . . . 333
gal_get_pol . . . 333
gal_get_roots . . . 333
gammamellininv . . . 371
gammamellininvinit . . . 371
gammamellininvrt . . . 371
gand . . . 237
garbage collecting . . . 16
gassoc_proto . . . 106
gaussred_from_QR . . . 194
gbezout . . . 241
gboundcf . . . 102
gcdext . . . 47
gcdii . . . 101
gceil . . . 234
gchari_lfun . . . 348
gclone . . . 25, 71, 72
gcloneref . . . 72
gclone_refc . . . 73
gcmp . . . 235
gcmpgs . . . 236
gcmpsg . . . 236
gcoeff . . . 15, 64, 283
gconj . . . 249
gcopy . . . 24, 25, 72
gcopy_avma . . . 71
gcopy_lg . . . 72
gcvtoi . . . 234
gcvtop . . . 224
gc_all . . . 20, 69
gc_all_unsafe . . . 70
gc_bool . . . 69
gc_const . . . 69
gc_double . . . 69
gc_GEN . . . 18, 19, 68
gc_INT . . . 68
gc_int . . . 69
gc_INT_affii . . . 68
gc_leaf . . . 70
gc_long . . . 69
gc_needed . . . 22
gc_NULL . . . 69
gc_slice . . . 70
gc_slice_unsafe . . . 70
gc_stack_update . . . 18, 71

gc_stoi 69
gc_ulong 69
gc_upto 19, 25, 70
gc_utoi 69
gc_utoipos 69
gdeuc 240
gdiv 243
gdivent 239
gdiventgs 239
gdiventres 239
gdiventsg 239
gdivexact 239
gdivgs 243
gdivgu 239, 243
gdivgunextu 239
gdivmod 240
gdivround 240
gdivsg 243
gdvd 239
gel 14, 15, 64, 283
GEN 13
GENbinbase 68
gener_F2xq 156
gener_Flxq 148
gener_FpXQ 130
gener_FpXQ_local 130
gener_Fq_local 131
GENtoGENstr 269
GENtoGENstr_nospace 269
GENtostr 37, 269
GENtostr_raw 269
GENtostr_unquoted 269
GENtoTeXstr 37, 269
gen_0 . . . . . 13, 30
gen_1 13
gen_2 13
gen_bkeval 219
gen_bkeval_powers 219
gen_cmp_RgX 239
gen_det 218
gen_digits 220
gen_ellgens 217, 357
gen_ellgroup 217
gen_factorback 245
gen_factored_order 216
gen_FpM_Wiedemann 197
gen_fromdigits 220
gen_Gauss 218
gen_Gauss_pivot 218
gen_gener 216
gen_indexsort 238
gen_indexsort_uniq 238
gen_ker 218
gen_m1 13
gen_m2 13
gen_matcolinvimage 218
gen_matcolmul 218
gen_matid 218
gen_matinvimage 218
gen_matmul 218
gen_order 216
gen_parapply 44
gen_parapply_percent 44
gen_parapply_slice 45
gen_parapply_slice_zv 45
gen_parpairwiseop 45
gen_PH_log 216
gen_plog 216
gen_Pollard_log 216
gen_pow 244
gen_powers 219, 245
gen_powu 244
gen_powu_fold 245
gen_powu_fold_i 245
gen_powu_i 244
gen_pow_fold 244, 245
gen_pow_fold_i 244, 245
gen_pow_i 244
gen_pow_init 245
gen_pow_table 245
gen_product 244
gen_RgX_bkeval 219
gen_search 238
gen_select_order 217
gen_setminus 238
gen_Shanks 216
gen_Shanks_init 216
gen_Shanks_log 216
gen_Shanks_sqrtn 216
gen_sort 237
gen_sort_inplace 238
gen_sort_shallow 238
gen_sort_uniq 238
gen_ZpM_Dixon 197
gen_ZpM_Dixon_Wiedemann 197
gen_ZpM_Newton 221
gen_ZpXX_Newton 221
gen_ZpX_Dixon 221

gen_ZpX_Newton . . . 221
geq . . . 237
gequal . . . 206, 235
gequal0 . . . 236
gequal1 . . . 236
gequalgs . . . 236
gequalm1 . . . 236
gequalsg . . . 236
gequalX . . . 235
gerepile . . . 68
getheap . . . 73
getrand . . . 102
getrealprecision . . . 258
gettime . . . 40
get_arith_Z . . . 217
get_arith_ZZM . . . 217
get_avma . . . 68
get_bnf . . . 299
get_bnfpol . . . 299
get_F2xqE_group . . . 217
get_F2xqX_degree . . . 158
get_F2xqX_mod . . . 158
get_F2xqX_var . . . 158
get_F2xq_field . . . 219
get_F2x_degree . . . 154
get_F2x_mod . . . 154
get_F2x_var . . . 154
get_FlxqE_group . . . 217
get_FlxqXQ_algebra . . . 220
get_FlxqX_algebra . . . 220
get_FlxqX_degree . . . 150
get_FlxqX_mod . . . 150
get_FlxqX_var . . . 150
get_Flxq_algebra . . . 220
get_Flxq_field . . . 218
get_Flxq_star . . . 217
get_Flx_algebra . . . 220
get_Flx_degree . . . 139
get_Flx_mod . . . 139
get_Flx_var . . . 139
get_Fl_red . . . 83
get_FpE_group . . . 217
get_FpXQE_group . . . 217
get_FpXQXQ_algebra . . . 220
get_FpXQX_algebra . . . 220
get_FpXQX_degree . . . 136
get_FpXQX_mod . . . 136
get_FpXQX_var . . . 136
get_FpXQ_algebra . . . 220
get_FpXQ_star . . . 217
get_FpX_algebra . . . 220
get_FpX_degree . . . 128
get_FpX_mod . . . 128
get_FpX_var . . . 128
get_Fp_field . . . 218
get_Fq_field . . . 218
get_lex . . . 291
get_modpr . . . 300
get_nf . . . 299
get_nfpol . . . 299
get_nf_field . . . 219
get_prid . . . 300
get_Rg_algebra . . . 220
gexpo . . . 28, 60
gexpo_safe . . . 60
gfloor . . . 234
gfrac . . . 234
ggamma1m1 . . . 260
ggcd . . . 241
gge . . . 237
ggt . . . 237
ghalf . . . 13
gidentical . . . 194, 235
gimag . . . 249
ginv . . . 243
ginvmod . . . 240
gisdouble . . . 223
gisexactzero . . . 235
glcm . . . 241
gle . . . 236
glt . . . 236
gmael . . . 15, 64
gmael1 . . . 15
gmael2 . . . 64
gmael3 . . . 64
gmael4 . . . 64
gmael5 . . . 64
gmax . . . 235
gmaxgs . . . 236
gmaxsg . . . 236
gmax_shallow . . . 235
gmin . . . 235
gmings . . . 236
gminsg . . . 236
gmin_shallow . . . 235
gmod . . . 240
gmodgs . . . 240
gmodsg . . . 240

gmodulgs 224
gmodulo 224
gmodulsg 224
gmodulss 224
gmul 243
gmul2n 234
gmulgs 243
gmulgu 243
gmulsg 243
gmulug 243
gne 237
gneg 242
gneg_i 242
gnorml1 . . . . . 245, 246
gnorml1_fake 246
gnorml2 245
gnot 237
gor 237
GP prototype 75
gphelp_keyword_list 57
gpinstall 56
gpow 243
gpowers 244
gpowgs 243
gprec 224
gprecision 61
gprec_w 224
gprec_wensure 224
gprec_wtrunc 224
gprimepi_lower_bound 180
gprimepi_upper_bound 180
gp_alarm_handler 57
gp_call 292
gp_call2 293
gp_callbool 293
gp_callprec 293
gp_callvoid 293
gp_context_restore 57
gp_context_save 57
gp_display_hist 57
gp_echo_and_log 57
gp_eval 292
gp_evalbool 292
gp_evalprec 292
gp_evalupto 292
gp_evalvoid 292
gp_filter 56
gp_format_prompt 57
gp_format_time 56
gp_handle_exception 55
gp_help 57
gp_load_gprc 56
gp_meta 56
gp_read_file 36, 56
gp_read_str 34, 35, 56, 78
gp_read_stream 36
gp_read_str_bitprec 36
gp_read_str_multiline 35
gp_read_str_prec 35
gp_sigint_fun 55
Gram matrix 191
gram_matrix 191
greal 249
gred_rfac2 31
grem 240
grndtoi 234
grootsof1 244
ground 234
groupelts_abelian_group 266
groupelts_center 266
groupelts_conjclasses 265
groupelts_conj_set 265
groupelts_exponent 266
groupelts_gen 265
groupelts_quotient 266
groupelts_set 265
groupelts_solvablesubgroups 266
groupelts_to_group 265
groupgen_export 267
group_abelianHNF 265
group_abelianSNF 266
group_domain 265
group_elts 265
group_export 266
group_ident 267
group_ident_trans 267
group_isA4S4 266
group_isabelian 265
group_istrivial 265
group_iswss 265
group_leftcoset 266
group_order 265
group_perm_normalize 266
group_quotient 266
group_rightcoset 266
group_set 265
group_subgroups 266
group_subgroup_isnormal 266

group_subgroup_is_faithful 265
group_to_cc 265
gshift 234
gsigne 28, 60
gsincos 260
gsizebyte 24
gsizeword 24
gsmith 343
gsmithall 343
gsprintf 270
gsqr 243
gsqrpowers 244
GSTR 32
gsub 243
gsubgs 243
gsubsg 243
gsubst 246
gsupnorm 246
gsupnorm_aux 246
gtocol 227
gtodouble 25, 223
gtofp 223, 224
gtolong 25, 223
gtomat 227
gtomp 224
gtopoly 225
gtopolyrev 225
gtos 223
gtoser 226
gtoser_prec 226, 227
gtou 223
gtovec 227
gtovecsmall 227
gtrans 283
gtrunc 234
gtrunc2n 90, 234
gunclone 25, 73
guncloneNULL 73
guncloneNULL_deep 73
gunclone_deep 73
gval 234
gvaluation 234
gvar 30, 33, 60
gvar2 61
gvsprintf 270
G_ZGC_mul 195
G_ZG_mul 195

H

halfgcdii 102
hammingu 81
hashentry 279
hashtable 279
hash_create 280
hash_create_GEN 280
hash_create_INT 280
hash_create_str 280
hash_create_ulong 280
hash_dbg 281
hash_destroy 281
hash_GEN 281
hash_haskey_GEN 281
hash_haskey_long 281
hash_init 280
hash_init_GEN 280
hash_init_ulong 280
hash_insert 280
hash_insert2 280
hash_insert_long 280
hash_keys 281
hash_keys_GEN 281
hash_remove 281
hash_remove_select 281
hash_search 280, 281
hash_search2 281
hash_select 281
hash_str 280, 281
hash_str_len 281
hash_values 281
hash_values_GEN 281
hash_zv 281
hclassno 335
hclassno6 335
hclassno6u 335
hclassno6u_no_cache 335
hclassnoF_fact 335
heap 14
Hermite_bound 344
hexadecimal tree 38
HIGHBIT 64
HIGHEXPOBIT 65
HIGHMASK 65
HIGHVALPBIT 65
HIGHWORD 65
hilbertii 108
hnf 342
hnfall 342
hnfdivide 340

hnflll 342
hnfmerge_get_1 315
hnfmod 342
hnfmodid 342
hnfperm 342
hnf_CENTER 339
hnf_divscale 340
hnf_invimage 340
hnf_invscale 340
hnf_MODID 339
hnf_PART 339
hnf_solve 340
hnf_Znproj 319
hqfeval 246
hyperell_locally_soluble 367
h_APROPOS 57
h_LONG 57
h_REGULAR 57

I

icopy 87
icopyifstack 72
icopyspec 88
icopy_avma 71
idealadd 314
idealaddmultoone 315
idealaddtoone 315
idealaddtoone_i 315
idealaddtoone_raw 315
idealappr 315, 333
idealappr0 333
idealapprfact 315
idealchinese 316
idealchineseinit 316
idealcoprime 315, 316
idealcoprimefact 316
idealdiv 314
idealdivexact 314
idealdivpowprime 314
idealfactor 313, 316, 317
idealfactor_limit 317
idealfactor_partial 317
idealfrobenius_aut 318
idealhnf 313, 314
idealhnf0 313
idealHNF_inv 315
idealHNF_inv_Z 315
idealHNF_mul 315
idealhnf_principal 314
idealhnf_shallow 314
idealhnf_two 314
idealHNF_Z_factor 317
idealHNF_Z_factor_i 317
idealinv 314
ideallog 313
ideallog_units 329
ideallog_units0 329
idealmoddivisor 326
idealmul 314
idealmulpowprime 314
idealmulred 314, 323
idealpow 314
idealpowred 314
idealpows 314
idealprimedec 313, 316
idealprimedec_degrees 316
idealprimedec_galois 316
idealprimedec_kummer 317
idealprimedec_limit_f 316
idealprimedec_limit_norm 316
idealprincipalunits 321
idealprod 314
idealprodprime 314
idealprodval 315
idealpseudomin 326
idealpseudominvec 326
idealpseudomin_nonscalar 326
idealpseudored 326
idealramfrobenius 318
idealramfrobenius_aut 318
idealramgroups_aut 318
idealred 326
idealred0 326
idealred_elt 326
idealsqr 314
idealstar 321
idealstar0 332
Idealstarmod 304
Idealstarprk 327
ideals_by_norm 315
idealtyp 300
identity_perm 263
identity_ZV 183
identity_zv 187
id_MAT 300
id_PRIME 300
id_PRINCIPAL 300

ifac_isprime 180
ifac_next 179
ifac_read 179
ifac_skip 179
ifac_start 179
iferrpari 290
ifpari 290
image 197
image2 197
imag_i 249
indexlexsort 237
indexpartial 323
indexsort 237
indexvecsort 237
indices_to_vec01 320
infinity 32
inf_get_sign 32
initprimetable 66
init_Flxq 142
init_Fq 137
inline closure 76
input 35
install 34, 39, 78, 79
int2n 86
int2u 86
int2um1 86
intcirc0 290
integer 26
integser 252
intfuncinit0 290
intnum0 290
int_LSW 27
int_MSW 27
int_nextW 27
int_normalize 27
int_precW 27
int_W 27
int_W_lg 27
invmod 104
invmod2BIL 82
invr 97
inv_content 242
isclone 26
iscomplex 236
isexactzero 235
isinexact 236
isinexactreal 236
isint 236
isint1 235
isintm1 235
isintzero 235
ismpzero 235
isonstack 72
isprime 181
isprimeAPRCL 181
isprimeECPP 181
isprimepower 107
isprincipal 324
isprincipalfact 324
isprincipalfact_or_fail 324
isprincipalforce 332
isprincipalgen 332
isprincipalgenforce 332
isprincipalraygen 332
isrationalzero 235
isrationalzeroscalar 236
isrealappr 236
issmall 236
is_357_power 107, 175
is_bigint 88
is_const_t 63
is_entry 74
is_extscalar_t 63
is_gchar_group 348
is_intreal_t 63
is_linit 369
is_matvec_t 63
is_nf_extfactor 300
is_nf_factor 300
is_noncalc_t 63
is_pm1 236
is_pth_power 175
is_qfb_t 63
is_rational_t 63
is_real_t 63
is_recursive_t 63
is_scalar_t 63
is_universal_constant 87
is_vec_t 63
is_Z_factor 178
is_Z_factornon0 178
is_Z_factorpos 178
itor 88
itor_lg 88
itos 25, 88, 223
itostr 269
itos_or_0 88
itou 88, 223

itou_or_0 88

K

killblock 72
krois 105
kroiu 105
Kronecker symbol 105
kronecker 105
Kronecker_to_F2xqX 158
Kronecker_to_FlxqX 150
Kronecker_to_FlxqX_pre 150
Kronecker_to_FpXQX 134
Kronecker_to_mod 214
Kronecker_to_ZXQX 203
Kronecker_to_ZXX 203
krosi 105
kross 105
kroui 105
krouu 105

L

laurentseries0 290
lcmii 101
ldata_get_an 369
ldata_get_conductor 369
ldata_get_degree 369
ldata_get_dual 369
ldata_get_gammavec 369
ldata_get_k 369
ldata_get_k1 369
ldata_get_residue 369
ldata_get_rootno 369
ldata_get_type 369
ldata_isreal 369
ldata_newprec 370
ldata_vecan 370
leading_coeff 30, 64
leafcopy 87, 283
leafcopy_avma 87
Legendre symbol 105
lexcmp 235
lexsort 237
lfun 370
lfunabelianrelinit 371
lfunellmfpeters 371
lfungchar 348
lfuninit 371
lfuninit_make 371
lfunlambda 371
lfunmisc_to_ldata 370
lfunmisc_to_ldata_shallow 370
lfunmisc_to_ldata_shallow_i 370
lfunprod_get_fact 369
lfunquadneg 371
lfunrootno 371
lfunrtopoles 370
lfunthetacheckinit 371
lfunthetacost 371
lfunzetakinit 371
lfun_get_bitprec 369
lfun_get_dom 369
lfun_get_domain 369
lfun_get_expot 369
lfun_get_factgammavec 369
lfun_get_k2 369
lfun_get_pol 369
lfun_get_Residue 369
lfun_get_step 369
lfun_get_w2 369
lg 26, 60
lg2prec 59
LGBITS 65
lgcols 61
lgefint 27, 60
LGnumBITS 65
lgpol 61
lg_increase 26
library mode 13
lift 225
lift0 225
liftall 225
liftall_shallow 225
liftint 225
liftint_shallow 225
liftpol 225
liftpol_shallow 225
lift_shallow 224
lincombii 97
lindep 344
lindep2 345
lindepfull_bit 345
lindep_bit 344
lindep_padic 345
lindep_Xadic 345
linit_get_ldata 369
linit_get_tech 369
linit_get_type 369

list . . . 32
LLL . . . 340, 343
lll . . . 343
lllfp . . . 344
lllgen . . . 343
lllgram . . . 343
lllgramgen . . . 343
lllgramint . . . 343
lllgramkerim . . . 343
lllgramkerimgen . . . 343
lllint . . . 343
lllintpartial . . . 344
lllintpartial_inplace . . . 344
lllkerim . . . 343
lllkerimgen . . . 343
LLL_ALL . . . 344
LLL_COMPATIBLE . . . 344
LLL_GRAM . . . 343
LLL_IM . . . 343, 344
LLL_INPLACE . . . 343, 344
LLL_KEEP_FIRST . . . 344
LLL_KER . . . 344
LOG10_2 . . . 66
LOG2_10 . . . 66
logint . . . 101
logintall . . . 101
logr_abs . . . 259
LONG_MAX . . . 64
long_to_rgb . . . 381
loop_break . . . 291
LOWMASK . . . 65
LOWWORD . . . 65

M

malloc . . . 274
mantissa2nr . . . 90
mantissa_real . . . 28, 90
map_proto_G . . . 106
map_proto_GL . . . 106
map_proto_lG . . . 106
map_proto_lGL . . . 106
matbrute . . . 272
matdet . . . 185
mathnf . . . 313
matid . . . 228
matid_F2m . . . 120
matid_F2xqM . . . 157
matid_Flm . . . 116
matid_FlxqM . . . 122
matpermanent . . . 185
matrix . . . 32
matrixqz . . . 342
matslice . . . 284
maxdd . . . 93
maxomegaoddu . . . 107
maxomegau . . . 107
maxprime . . . 13, 66
maxprimelim . . . 66
maxprimeN . . . 66
maxprime_check . . . 66
maxss . . . 93
maxuu . . . 93
MAXVARN . . . 65
MEDDEFAULTPREC . . . 16, 64
merge_factor . . . 238
merge_sort_uniq . . . 238
mfcharmodulus . . . 377
mfcharorder . . . 377
mfcharpol . . . 377
mfcuspdim . . . 377
MFcusp_get_vMjd . . . 376
mfdiv_val . . . 377
mfeisensteindim . . . 377
mfeisensteinspaceinit . . . 377
mfembed . . . 377
mffulldim . . . 377
mfiscuspidal . . . 377
mfmatembed . . . 377
mfnewdim . . . 377
MFnew_get_vj . . . 376
mfnumcuspsu . . . 373
mfnumcuspsu_fact . . . 373
mfnumcusps_fact . . . 373
mfolddim . . . 377
mfsturmNgk . . . 377
mfsturmNk . . . 377
mfsturm_mf . . . 377
mftobasisES . . . 377
mftocol . . . 377
mfvecembed . . . 377
mfvectomat . . . 377
MF_get_basis . . . 375, 376
MF_get_CHI . . . 375
mf_get_CHI . . . 376
MF_get_dim . . . 375
MF_get_E . . . 375
mf_get_field . . . 376

MF_get_fields . . . 375
MF_get_gk . . . 375
mf_get_gk . . . 376
MF_get_gN . . . 375
mf_get_gN . . . 376
MF_get_k . . . 375
mf_get_k . . . 376
MF_get_M . . . 376
MF_get_Mindex . . . 376
MF_get_Minv . . . 376
MF_get_N . . . 375
mf_get_N . . . 376
MF_get_newforms . . . 375
mf_get_NK . . . 376
MF_get_r . . . 375
mf_get_r . . . 376
MF_get_S . . . 375, 376
MF_get_space . . . 375
mf_get_type . . . 376
millerrabin . . . 181
mindd . . . 93
minss . . . 93
minuu . . . 93
mkcol . . . 230
mkcol2 . . . 230
mkcol2s . . . 229
mkcol3 . . . 230
mkcol3s . . . 229
mkcol4 . . . 230
mkcol4s . . . 229
mkcol5 . . . 230
mkcol6 . . . 230
mkcolcopy . . . 229
mkcoln . . . 24, 233
mkcols . . . 229
mkcomplex . . . 230
mkerr . . . 231
mkfrac . . . 230
mkfraccopy . . . 229
mkfracss . . . 228
mkintmod . . . 230
mkintmodu . . . 228
mkintn . . . 88, 233
mkmat . . . 230
mkmat2 . . . 230
mkmat22 . . . 230
mkmat22s . . . 229
mkmat3 . . . 231
mkmat4 . . . 231
mkmat5 . . . 231
mkmatcopy . . . 229
mkmoo . . . 32
mkoo . . . 32
mkpadic . . . 230
mkpolmod . . . 230
mkpoln . . . 24, 233
mkqfb . . . 231
mkquad . . . 230
mkrfrac . . . 230
mkrfraccopy . . . 229
mkvec . . . 231
mkvec2 . . . 231
mkvec2copy . . . 229
mkvec2s . . . 229
mkvec3 . . . 231
mkvec3s . . . 229
mkvec4 . . . 231
mkvec4s . . . 229
mkvec5 . . . 231
mkvec5s . . . 229
mkvec6 . . . 231
mkvec6s . . . 229
mkveccopy . . . 229
mkvecn . . . 24, 233
mkvecs . . . 229
mkvecsmall . . . 229
mkvecsmall2 . . . 229
mkvecsmall3 . . . 229
mkvecsmall4 . . . 229
mkvecsmall5 . . . 229
mkvecsmalln . . . 229
Mod16 . . . 100
mod16 . . . 100
Mod2 . . . 100
mod2 . . . 100
mod2BIL . . . 100
Mod32 . . . 100
mod32 . . . 100
Mod4 . . . 100
mod4 . . . 100
Mod64 . . . 100
mod64 . . . 100
Mod8 . . . 100
mod8 . . . 100
modinv_good_disc . . . 367
modinv_good_prime . . . 367
modinv_height_factor . . . 367
modinv_is_double_eta . . . 367

modinv_is_Weber 367
modpr_genFq 318
modpr_get_p 318
modpr_get_pr 318
modpr_get_T 318
modreverse 214
modRr_i 249
modRr_safe 249
moebiusu 106
moebiusu_fact 106
monomial_F2x 155
monomial_Flx 142
mpabs 92
mpabs_shallow 93
mpacos 258
mpacosh 259
mpadd 15
mpasin 258
mpasinh 259
mpatan 258
mpatanh 259
mpbern 263
mpceil 89
mpcmp 93
mpcopy 87
mpcos 258
mpcosh 258
mpcotan 258
mpcotanh 259
mpeint1 259
mpeuler 263
mpexp 259
mpexpm1 259
mpexpo 60
mpfloor 89
mplambertW 259
mplambertX 259
mplambertxlogx_x 259
mplambertx_logx 259
mplog 259
mplog1p 259
mplog2 263
mpneg 93
mpodd 100
mppi 262
mpprimorial 181
mpround 89
mpshift 89
mpsin 258
mpsinc 258
mpsincos 260
mpsincosm1 259
mpsinh 258
mpsinhcosh 260
mpsqr 93
mptan 258
mptanh 259
mptrunc 89
mpveceint1 259
mseval2_ooQ 373
msgtimer 40
mspadic_parse_chi 373
mspadic_unit_eigenvalue 373
mulcxI 230
mulcxmI 230
mulcxpowIs 230
mulimag 249
muliu 96
mulll 81
mulreal 249
mulsubii 97
muls_interval 97
multable 310
mului 96
muluu 96
muluui 96
mulu_interval 97
mulu_interval_step 97
mul_content 242
mul_denom 242
M_LN2 65
M_PI 65

N

name_numerr 279
name_var 34, 74
nbits2extraprec 60
nbits2lg 59
nbits2ndec 59
nbits2nlong 59
nbits2prec 59
nbrows 61
nchar2nlong 59
ncharvecexpo 348
ncV_chinese_center 164
ncV_chinese_center_tree 165
ndec2nbits 60

ndec2nlong 59
ndec2prec 59
negi 92
negr 93
newblock 72
new_chunk 67
new_chunk_resize 67
nfadd 307
nfalgtobasis 308
nfarchstar 321
nfbasistoalg 308
nfchecksigns 320
nfcyclotomicunits 332
nfC_multable_mul 309
nfC_nf_mul 309
nfdiv 308
nfdiveuc 308
nfdivrem 308
nfeltembed_i 321
nfeltup 330
nfembed 321
nffactorback 313
nfgaloisconj 331
nfgaloismatrix 334
nfgaloismatrixapply 334
nfgaloispermtobasis 333
nfgcd 205
nfgcd_all 205
nfgwkummer 330
nfinit_basic 322
nfinit_complete 323
nfinv 308
nfinvmodideal 309
nfissquarefree 330
nflogembed 322
nfmaxord 322
nfmaxord_t 322, 323
nfmaxord_to_nf 322, 323
nfmod 308
nfmodprinit 318, 319
nfmul 308
nfmuli 309
nfM_det 310
nfM_inv 310
nfM_ker 310
nfM_mul 310
nfM_nfC_mul 310
nfM_to_FqM 319
nfnewprec 307
nfnewprec_shallow 307
nfnorm 308
nfpoleval 308
nfpow 308
nfpowmodideal 309
nfpow_u 308
nfrootsof1 332
nfroots_if_split 331
nfsign 313, 320
nfsign_arch 313, 320
nfsign_from_logarch 321
nfsign_fu 320
nfsign_tu 321
nfsign_units 320
nfsqr 308
nfsqri 309
nfsub 308
nftrace 308
nftyp 299
nfval 308, 319
nfV_cxlog 322
nfV_idealhnf 314
nfV_to_FqV 319
nfV_to_scalar_or_alg 308
nfX_disc 309
nfX_resultant 309
nfX_to_FqX 319
nfX_to_monic 309
nf_cxlog 321
nf_cxlog_normalize 321
nf_deg1_prime 317
nf_FORCE 324
nf_GEN 323
nf_GENMAT 323, 324
nf_GEN_IF_PRINCIPAL 324
nf_get_allroots 301
nf_get_degree 301
nf_get_diff 301
nf_get_disc 301
nf_get_G 301
nf_get_Gtwist 325, 326, 344
nf_get_Gtwist1 325
nf_get_index 301
nf_get_invzk 301
nf_get_M 301, 324
nf_get_pol 301
nf_get_prec 301
nf_get_r1 301
nf_get_r2 301

nf_get_ramified_primes 301
nf_get_roots 301
nf_get_roundG 301, 325, 326
nf_get_sign 301
nf_get_Tr 301
nf_get_varn 301
nf_get_zk 301
nf_get_zkden 301
nf_get_zkprimpart 301
nf_hyperell_locally_soluble 367
nf_nfzk 330
nf_PARTIALFACT 322
nf_pV_to_prV 317
nf_rnfeq 330
nf_rnfeqsimple 330
nf_ROUND2 322
nf_to_Fp_coprime 327
nf_to_Fq 318
nf_to_Fq_init 318
nf_to_scalar_or_alg 308
nf_to_scalar_or_basis 308
nf_to_scalar_or_polmod 308
nmV_chinese_center 164
nmV_chinese_center_tree 165
nm_Z_mul 173
nonsquare_Fl 83
normalized character 346
normalizepol 31, 225
normalizepol_approx 225
normalizepol_lg 225
normalizeser 227
normalize_frac 63
NO_VARIABLE 30, 33, 61, 65
numberofconjugates 331
numdivu 107
numdivu_fact 107
numerr_name 279
numer_i 241
nv_fromdigits_2k 91
nxCV_chinese_center 164
nxCV_chinese_center_tree 165
nxMV_chinese_center 164
nxV_chinese_center 164
nxV_chinese_center_tree 165

O

obj 296
obj_check 296
obj_checkbuild 297
obj_checkbuild_padicprec 297
obj_checkbuild_prec 297
obj_checkbuild_realprec 297
obj_free 297, 355
obj_init 296
obj_insert 297
obj_insert_shallow 297
obj_reinit 296
odd 82
odd_prime_divisors 177
omega 178
omegau 107
ONLY_DIVIDES 110, 210
ONLY_REM 110, 209
outmat 37
output 36
output 37, 38, 39, 272
out_printf 271
out_putc 271
out_puts 271
out_term_color 272
out_vprintf 271

P

p-adic number 29
padicprec 170
padicprec_relative 170
padic_p 29
padic_pd 29
padic_to_Fl 171
padic_to_Fp 113
padic_to_Q 170
padic_to_Q_shallow 170
padic_u 29
parfor 43
parforeach 43
parforeach_init 43
parforeach_next 43
parforeach_stop 43
parforprime 44
parforprimestep 44
parforprimestep_init 44
parforprime_init 44
parforprime_next 44
parforprime_stop 44
parforstep 44
parforstep_init 44

parforstep_next 44
parforstep_stop 44
parforvec 44
parforvec_init 44
parforvec_next 44
parforvec_stop 44
parfor_init 43
parfor_next 43
parfor_stop 43
paricfg_buildinfo 80
paricfg_compiledate 80
paricfg_datadir 80
paricfg_gphelp 80
paricfg_mt_engine 80
paricfg_vcsversion 80
paricfg_version 80
paricfg_version_code 80
pariErr 271
PariOUT 270
pariOut 271
pariplot0 290
paristack_newrsize 54
paristack_resize 53
paristack_setsize 53
parivstack_reset 54
parivstack_resize 54
pari_add_defaults_module 54
pari_add_function 54
pari_add_hist 57
pari_add_module 54
pari_alarm 57
pari_ask_confirm 56
pari_base64 270
pari_breakpoint 290
pari_calloc 16
pari_CATCH 45
pari_CATCH_reset 46
pari_center 56
pari_close 51
pari_close_opts 53
pari_community 56
pari_compile_str 56
pari_daemon 53
pari_ENDCATCH 46
pari_err 32, 37, 38, 46, 274, 295
pari_err2str 279
pari_errfile 271
pari_err_display 279
pari_err_last 46
pari_err_TYPE 354
pari_fclose 273
pari_flush 36, 271
pari_fopen 273
pari_fopengz 273
pari_fopen_or_fail 273
pari_fprintf 37
pari_fread_chars 272
pari_free 16, 67
pari_get_hist 57
pari_get_histrtime 57
pari_get_histtime 57
pari_get_homedir 273
pari_histtime 57
pari_hit_return 56
pari_infile 56
pari_init 13, 14, 51
pari_init_opts 51
pari_init_primes 52, 53
pari_is_default 293
pari_is_dir 272
pari_is_file 272
pari_kernel_close 52
pari_kernel_init 52
pari_kernel_version 80
pari_kill_plot_engine 379
pari_last_was_newline 271
pari_library_path 56
pari_malloc 16, 67, 277
pari_mt_close 53
pari_mt_init 52
pari_nb_hist 57
PARI_OLD_NAMES 14
pari_outfile 36, 271
PARI_plot 379
pari_plot_by_file 381
pari_PRIMES 14
pari_printf 36, 38, 75, 271, 272
pari_print_version 56
pari_putc 36, 75, 271
pari_puts 36, 75, 271, 272
pari_rand 102
pari_realloc 16, 277
pari_realloc_ip 16
pari_RETRY 46
pari_safefopen 273
pari_set_last_newline 271
pari_set_plot_engine 379
pari_sighandler 53

pari_sig_init 53
pari_sp 17
pari_sprintf 37, 269
pari_stackcheck_init 53
pari_stack_alloc 282
pari_stack_base 283
pari_stack_delete 283
pari_stack_init 282
pari_stack_new 282
pari_stack_pushp 283
pari_stdin_isatty 273
pari_str 270
pari_strdup 269
pari_strndup 269
pari_thread_alloc 385
pari_thread_close 385
pari_thread_free 385
pari_thread_init 385
pari_thread_start 385
pari_thread_valloc 385
pari_timer 39
pari_TRY 46
pari_unique_dir 274
pari_unique_filename 274
pari_unique_filename_suffix 274
pari_unlink 272
pari_var_close 74
pari_var_create 74
pari_var_init 74
pari_var_next 74
pari_var_next_temp 74
PARI_VERSION 80
pari_version 80
PARI_VERSION_SHIFT 80
pari_vfprintf 37
pari_vprintf 37
pari_vsprintf 37
pari_warn 38
path_expand 273
perm_commute 263
perm_conj 263
perm_cycles 264
perm_inv 263
perm_mul 263
perm_order 264
perm_orderu 264
perm_pow 263
perm_powu 263
perm_sign 264
perm_sqr 263
perm_to_GAP 264
perm_to_Z 264
pgener_Fl 83
pgener_Fl_local 83
pgener_Fp 105
pgener_Fp_local 105
pgener_Zl 83
pgener_Zp 105
Pi2n 263
PiI2 263
PiI2n 263
plotbox 379
plotclip 379
plotcolor 379
plotcopy 379
plotcursor 379
plotdraw 379
ploth 379
ploth0 290
plothexport0 291
plothraw 379
plothsizes 379
plotinit 379
plotkill 379
plotline 379
plotlines 379
plotlinetype 380
plotmove 380
plotpoints 380
plotpointsize 380
plotpointtype 380
plotrbox 380
plotrecth 379
plotrecthraw 380
plotrline 380
plotrmove 380
plotrpoint 380
plotscale 380
plotstring 380
point_to_a4a6 355
point_to_a4a6_Fl 355
pol0_F2x 154
pol0_Flx 142
pol1_F2x 155
pol1_F2xX 157
pol1_Flx 142
pol1_FlxX 148
polclass 366

polcoef_i . . . 251
poldivrem . . . 240
poleval . . . 209, 246
polgalois . . . 267
polhensellift . . . 167, 168
polintspec . . . 191
polint_i . . . 191
pollegendre_reduced . . . 252
polmod . . . 30
polmodular . . . 366
polmodular_ZM . . . 367
polmodular_ZXX . . . 366
polmod_nffix . . . 331
polmod_nffix2 . . . 331
polmod_to_embed . . . 252
Polred . . . 333
polred0 . . . 333
polredabs . . . 333
polredabs2 . . . 333
polredabsall . . . 333
Polrev . . . 227
polxn_Flx . . . 142
polx_F2x . . . 155
polx_F2xX . . . 157
polx_Flx . . . 142
polx_FlxX . . . 148
polx_zx . . . 205
polynomial . . . 30
pol_0 . . . 227
pol_1 . . . 227
pol_x . . . 227
pol_xn . . . 227
pol_xnall . . . 227
pol_x_powers . . . 227
pop_lex . . . 77, 291
pow2Pis . . . 260
powcx . . . 260
powcx_prec . . . 260
power series . . . 31
powersr . . . 100
powgi . . . 244
powii . . . 100
powis . . . 100
powIs . . . 101
powiu . . . 100
powPis . . . 260
powrfrac . . . 100
powrs . . . 100
powrshalf . . . 100
powru . . . 100
powruhalf . . . 100
powuu . . . 100
ppg . . . 178
ppi . . . 178
pple . . . 178
ppo . . . 178
prec2lg . . . 60
prec2nbits . . . 60
prec2nbits_mul . . . 60
prec2ndec . . . 60
precdbl . . . 59
precision . . . 61
precision0 . . . 224
precp . . . 29, 60
PRECPBITS . . . 65
PRECPSHIFT . . . 65
preferences file . . . 78
prid . . . 316
prime . . . 180
primeform . . . 337
primeform_u . . . 337
primepi_lower_bound . . . 180
primepi_upper_bound . . . 180
primes . . . 180
primes_interval . . . 180
primes_interval_zv . . . 180
PRIMES_search . . . 66
primes_upto_zv . . . 181
primes_zv . . . 180
prime_fact . . . 228
prime_over_p_full . . . 316
primitive root . . . 83
primitive_part . . . 241
primpart . . . 241
printf . . . 36, 37, 75
print_fun_list . . . 57
prodeuler0 . . . 291
prodinf0 . . . 291
prodprimes . . . 66
prodprimeslim . . . 66
prototype code . . . 78
prV_lcm_capZ . . . 317
prV_primes . . . 317
pr_anti_uniformizer . . . 317
pr_basis_perm . . . 319
pr_equal . . . 320
pr_get_e . . . 316
pr_get_f . . . 316

pr_get_gen . . . 316
pr_get_p . . . 316
pr_get_tau . . . 316
pr_hnf . . . 316
pr_inv . . . 316
pr_inv_p . . . 316
pr_is_inert . . . 316
pr_norm . . . 316
pr_uniformizer . . . 317
psdraw . . . 380
psi1series . . . 252
psploth . . . 380
psplothraw . . . 380
pthread_join . . . 385
push_lex . . . 77, 291
putch . . . 271
puts . . . 271
p_to_FF . . . 253
p_to_FF(p,0) . . . 253

Q

QabM_tracerel . . . 311
QabV_tracerel . . . 311
Qab_tracerel . . . 311
Qab_trace_init . . . 311
Qdivii . . . 241
Qdivis . . . 241
Qdiviu . . . 241
Qevproj_apply . . . 187
Qevproj_apply_vecei . . . 187
Qevproj_down . . . 187
Qevproj_init . . . 187
qfbcomp . . . 336
qfbcompraw . . . 336
qfbcompraw_i . . . 336
qfbcomp_i . . . 336
qfbforms . . . 251
qfbpow . . . 336
qfbpowraw . . . 336
qfbpows . . . 336
qfbpow_i . . . 336
qfbred . . . 335
qfbred_i . . . 335
qfbsolve . . . 336
qfbsqr . . . 336
qfbsqr_i . . . 336
qfb_1 . . . 335
qfb_disc . . . 251
qfb_disc3 . . . 251
qfb_equal1 . . . 335
qfb_ZM_apply . . . 251
qfeval . . . 246
qfevalb . . . 246
qfiseven . . . 192
qfisolvep . . . 336
qfi_log . . . 336
qfi_order . . . 336
qfi_red . . . 335
qfi_Shanks . . . 336
qflll0 . . . 343
qflllgram0 . . . 343
qfr3 . . . 337
qfr3_comp . . . 337
qfr3_compraw . . . 337
qfr3_pow . . . 338
qfr3_red . . . 338
qfr3_rho . . . 338
qfr3_to_qfr . . . 338
qfr5 . . . 337
qfr5_comp . . . 338
qfr5_compraw . . . 338
qfr5_dist . . . 338
qfr5_pow . . . 338
qfr5_red . . . 338
qfr5_rho . . . 338
qfr5_to_qfr . . . 338
qfrsolvep . . . 337
qfr_boundcf . . . 336
qfr_cf . . . 335
qfr_data_init . . . 337
qfr_red . . . 335
qfr_to_qfr5 . . . 338
qf_RgM_apply . . . 246
qf_ZM_apply . . . 246
QM_charpoly_ZX . . . 186, 187
QM_charpoly_ZX_bound . . . 186
QM_det . . . 189
QM_gauss . . . 187
QM_gauss_i . . . 187
QM_image . . . 187
QM_image_shallow . . . 187
QM_ImQ . . . 342
QM_ImQ_all . . . 342
QM_ImQ_hnf . . . 342
QM_ImQ_hnfall . . . 342
QM_ImZ . . . 342
QM_ImZ_all . . . 342

QM_ImZ_hnf . . . 342
QM_ImZ_hnfall . . . 342
QM_indexrank . . . 187
QM_inv . . . 187
QM_ishnf . . . 187
QM_ker . . . 189
QM_minors_coprime . . . 342
QM_mul . . . 189
QM_QC_mul . . . 189
QM_rank . . . 187
QM_sqr . . . 189
QpV_to_QV . . . 170
Qp_agm2_sequence . . . 262
Qp_ascending_Landen . . . 262
Qp_descending_Landen . . . 262
Qp_exp . . . 261
Qp_exp_prec . . . 261
Qp_gamma . . . 261
Qp_log . . . 262
Qp_psi . . . 262
Qp_sqrt . . . 262
Qp_sqrtn . . . 262
Qp_zeta . . . 261
Qp_zetahurwitz . . . 261
QR_init . . . 194
Qtoss . . . 228
quadclassno . . . 334
quadclassnoF . . . 334
quadclassnoF_fact . . . 335
quadclassnos . . . 334
quadnorm . . . 251
quadpoly . . . 29
quadpoly_i . . . 230
quadratic number . . . 29
quadratic_prec_mask . . . 170
quadtofp . . . 223
quad_disc . . . 251
quasi-normalized character . . . 346
quotient_group . . . 266
quotient_groupelts . . . 266
quotient_perm . . . 266
quotient_subgroup_lift . . . 266
QV_isscalar . . . 192
QXk_gcd . . . 203
QXQC_to_mod_shallow . . . 161
QXQM_mul . . . 205
QXQM_sqr . . . 205
QXQM_to_mod_shallow . . . 161
QXQV_to_FpM . . . 319
QXQV_to_mod . . . 161
QXQXV_to_mod . . . 161
QXQX_gcd . . . 205
QXQX_homogenous_evalpow . . . 205
QXQX_mul . . . 205
QXQX_powers . . . 205
QXQX_QXQ_mul . . . 205
QXQX_sqr . . . 205
QXQX_to_mod_shallow . . . 161
QXQ_charpoly . . . 204
QXQ_div . . . 204
QXQ_intnorm . . . 204
QXQ_inv . . . 204
QXQ_mul . . . 204
QXQ_mul(A,QXQ_inv(B,T),T) . . . 204
QXQ_norm . . . 204
QXQ_powers . . . 204
QXQ_reverse . . . 204
QXQ_sqr . . . 204
QXQ_to_mod_shallow . . . 161
QXV_QXQ_eval . . . 205
QXY_QXQ_evalx . . . 205
QX_complex_roots . . . 204, 252
QX_disc . . . 204
QX_factor . . . 204
QX_gcd . . . 204
QX_mul . . . 203
QX_resultant . . . 204
QX_sqr . . . 203
QX_ZXQV_eval . . . 205
QX_ZX_rem . . . 204
Q_abs . . . 241
Q_abs_shallow . . . 241
Q_content . . . 242
Q_content_safe . . . 242
Q_denom . . . 242
Q_denom_safe . . . 242
Q_div_to_int . . . 242
Q_factor . . . 178
Q_factor_limit . . . 178
Q_gcd . . . 241
Q_lval . . . 241
Q_lvalrem . . . 241
Q_muli_to_int . . . 242
Q_mul_to_int . . . 242
Q_primitive_part . . . 242
Q_primpart . . . 242
Q_pval . . . 241
Q_pvalrem . . . 241

Q_remove_denom 242, 307
Q_to_famat 312

R

radicalu 106
random 102
randomi 102
randomr 102
random_bits 102
random_F2x 155
random_F2xqE 363
random_F2xqX 158
random_Fl 83, 102
random_Fle 362
random_Fle_pre 362
random_Flj_pre 362
random_Flv 116
random_Flx 143
random_FlxqE 364
random_FlxqX 150
random_FpC 113
random_FpE 360
random_FpV 113
random_FpX 127
random_FpXQE 366
random_FpXQX 134
random_Fq 129
random_zv 187
rational function 31
rational number 29
raw 269
rcopy 87
rdivii 98
rdiviiz 98
rdivii_lg 98
rdivis 98
rdivsi 98
rdivss 98
read 35, 36
readseq 36
real number 28
real2n 86
realprec 28, 59
real_0 86
real_0_expo 86
real_1 86
real_i 249
real_m1 86
real_m2n 86
rect2ps 380
rect2ps_i 380
rect2svg 381
reducemodinvertible 345
reducemodlll 346
remi2n 97, 198
remlll_pre 84
reml_pre 84
remsBIL 65
residual_characteristic 251
resultant (reduced) 167
resultant 240, 251
resultant2 252
retconst_col 232
retconst_vec 231
retgc_const 69
retmkcol 232
retmkcol2 232
retmkcol3 232
retmkcol4 232
retmkcol5 232
retmkcol6 232
retmkcomplex 232
retmkfrac 232
retmkintmod 232
retmkmat 232
retmkmat2 232
retmkmat22 232
retmkmat3 232
retmkmat4 232
retmkmat5 232
retmkpadic 232
retmkpadic_i 232
retmkpolmod 232
retmkqfb 232
retmkquad 232
retmkrfrac 232
retmkvec 231
retmkvec2 231
retmkvec3 231
retmkvec4 231
retmkvec5 231
retmkvec6 231
rfracrecip 207
rfracrecip_to_ser_absolute 226
rfrac_deflate 207
rfrac_deflate_max 207
rfrac_deflate_order 207

rfrac_to_ser 226
rfrac_to_ser_i 226
RgC_add 189
RgC_fpnorml2 193
RgC_gtofp 192
RgC_gtomp 192
RgC_is_ei 192
RgC_is_FFC 254
RgC_neg 189
RgC_RgM_mul 190
RgC_RgV_mul 190
RgC_RgV_mulrealsym 190
RgC_Rg_add 189
RgC_Rg_div 190
RgC_Rg_mul 190
RgC_Rg_sub 189
RgC_sub 189
RgC_to_FpC 113
RgC_to_FqC 115
RgC_to_nfC 309
RgE_is_FpE 361
RgE_is_FpXQE 366
RgE_to_F2xqE 364
RgE_to_FlxqE 365
RgE_to_FpE 361
RgE_to_FpXQE 366
RgMrow_RgC_mul 190
RgMrow_zc_mul 173
RgMs_structelim 196
RgM_add 189
RgM_Babai 194
RgM_check_ZM 184
RgM_Cholesky 194
RgM_det_triangular 193
RgM_diagonal 192
RgM_diagonal_shallow 192
RgM_dimensions 189
RgM_div 190
RgM_flatten_RgC 284
RgM_flatten_RgV 284
RgM_fpnorml2 . . . . . 193, 245
RgM_Fp_init 113
RgM_gram_schmidt 194
RgM_gtofp 193
RgM_gtomp 193, 194
RgM_Hadamard 193
RgM_hnfall 343
RgM_inv 193
RgM_invimage 193
RgM_inv_upper 193
RgM_isdiagonal 192
RgM_isidentity 192
RgM_isscalar 192
RgM_is_FFM 254
RgM_is_FpM 112
RgM_is_QM 192
RgM_is_ZM 192
RgM_minor 284
RgM_mul 190
RgM_mulreal 190
RgM_multosym 190
RgM_neg 189
RgM_powers 190
RgM_QR_init 194
RgM_rescale_to_int 184
RgM_RgC_invimage 193
RgM_RgC_mul 190
RgM_RgC_type 112
RgM_RgV_mul 190
RgM_RgX_mul 190
RgM_Rg_add 189
RgM_Rg_add_shallow 189
RgM_Rg_div 190
RgM_Rg_mul 190
RgM_Rg_sub 189
RgM_Rg_sub_shallow 189
RgM_shallowcopy 283
RgM_solve 193
RgM_solve_realimag 193
RgM_sqr 190
RgM_sub 189
RgM_sumcol 191
RgM_to_F2m 120
RgM_to_F3m 122
RgM_to_Flm 172
RgM_to_FpM 113
RgM_to_FqM 115
RgM_to_nfM 309
RgM_to_RgXV 226
RgM_to_RgXV_reverse 226
RgM_to_RgXX 226
RgM_transmul 190
RgM_transmultosym 190
RgM_type 112
RgM_type2 112
RgM_zc_mul 173
RgM_zm_mul 173
RgM_ZM_mul 190

RgV_add 189
RgV_check_ZV 182
RgV_dotproduct 191
RgV_dotsquare 191
RgV_F2v_extract_shallow 284
RgV_gtofp 192
RgV_isin 192
RgV_isin_i 192
RgV_isscalar 192
RgV_is_arithprog 182
RgV_is_FpV 112
RgV_is_prV 300
RgV_is_QV 182
RgV_is_ZMV 188
RgV_is_ZV 182
RgV_is_ZVnon0 182
RgV_is_ZVpos 182
RgV_kill0 191
RgV_neg 189
RgV_nffix 331
RgV_polint 191
RgV_prod 190
RgV_RgC_mul 190
RgV_RgM_mul 190
RgV_Rg_mul 190
RgV_sub 189
RgV_sum 190
RgV_sumpart 190
RgV_sumpart2 191
RgV_to_F2v 120
RgV_to_F3v 121
RgV_to_Flv 172
RgV_to_FpV 113
RgV_to_RgM 226
RgV_to_RgX 225
RgV_to_RgX_reverse 225
RgV_to_ser 226
RgV_to_str 269, 270
RgV_type 112
RgV_type2 112
RgV_zc_mul 173
RgV_zm_mul 173
RgXnV_red_shallow 213
RgXn_div 212
RgXn_div_i 212
RgXn_eval 213
RgXn_exp 213
RgXn_expint 213
RgXn_inv 212
RgXn_inv_i 212
RgXn_mul 212
RgXn_powers 212
RgXn_powu 212
RgXn_powu_i 212
RgXn_recip_shallow 212
RgXn_red_shallow 212
RgXn_reverse 213
RgXn_sqr 212
RgXn_sqrt 212
RgXQC_red 214
RgXQM_mul 214
RgXQM_red 214
RgXQV_factorback 214
RgXQV_red 214
RgXQV_RgXQ_mul 214
RgXQX_div 214
RgXQX_divrem 214
RgXQX_mul 214
RgXQX_powers 214
RgXQX_pseudodivrem 210
RgXQX_pseudorem 210
RgXQX_red 214
RgXQX_rem 214
RgXQX_RgXQ_mul 214
RgXQX_RgXQ_translate 214
RgXQX_sqr 214
RgXQ_charpoly 213
RgXQ_inv 213
RgXQ_matrix_pow 213
RgXQ_minpoly 213
RgXQ_mul 213
RgXQ_norm 213
RgXQ_pow 213
RgXQ_powers 213
RgXQ_powu 213
RgXQ_ratlift 213
RgXQ_reverse 214
RgXQ_sqr 213
RgXQ_trace 213
RgXV_maxdegree 208
RgXV_prod 209
RgXV_rescale 211
RgXV_RgV_eval 209
RgXV_to_FlxV 171
RgXV_to_RgM 226
RgXV_unscale 211
RgXX_to_Kronecker 134, 203, 207, 208
RgXX_to_Kronecker_spec 208

RgXX_to_Kronecker_var 208
RgXX_to_RgM 226
RgXY_cxevalx 246
RgXY_degreex 226
RgXY_derivx 226
RgXY_swap 226
RgXY_swapspec 226
RgX_act_Gl2Q 212
RgX_act_ZGl2Q 212
RgX_add 208
RgX_addmulXn 210
RgX_addmulXn_shallow 210
RgX_addspec 210
RgX_addspec_shallow 210
RgX_affine 211
RgX_blocks 143, 149, 206
RgX_check_QX 203
RgX_check_ZX 197
RgX_check_ZXX 203
RgX_chinese_coprime 211
RgX_coeff 206
RgX_copy 206
RgX_cxeval 246
RgX_deflate 207
RgX_deflate_max 207
RgX_deflate_order 207
RgX_degree 206
RgX_deriv 211
RgX_derivn 211
RgX_digits 210
RgX_disc 211
RgX_div 210
RgX_divrem 209
RgX_divs 209
RgX_div_by_X_x 210
RgX_equal 206
RgX_equal_var 206
RgX_even_odd 155, 206
RgX_extgcd 211
RgX_extgcd_simple 211
RgX_fpnorml2 211
RgX_gcd 210, 211
RgX_gcd_simple 211
RgX_Graeffe 207
RgX_gtofp 211
RgX_halfgcd 211
RgX_halfgcd_all 211
RgX_homogenize 207
RgX_homogenize_deg 207
RgX_homogenous_derivn 212
RgX_homogenous_eval 212
RgX_homogenous_evalpow 212
RgX_inflate 207
RgX_integ 211
RgX_isscalar 206
RgX_is_FpX 123
RgX_is_FpXQX 128
RgX_is_monomial 206
RgX_is_QX 206
RgX_is_rational 206
RgX_is_ZX 206
RgX_is_ZXX 206
RgX_mul 209
RgX_mul2n 209
RgX_mulhigh_i 212
RgX_muls 209
RgX_mulspec 210
RgX_mulXn 210
RgX_mul_i 209
RgX_mul_normalized 209
RgX_neg 208
RgX_nffix 331
RgX_normalize 209
RgX_pseudodivrem 210
RgX_pseudorem 210
RgX_recip 207
RgX_recip_i 207
RgX_recip_shallow 207
RgX_rem 210
RgX_renormalize 206
RgX_renormalize_lg 206
RgX_rescale 211
RgX_rescale_to_int 207
RgX_resultant_all 211
RgX_RgMV_eval 246
RgX_RgM_eval 246
RgX_RgV_eval 209
RgX_RgXnV_eval 213
RgX_RgXn_eval 213
RgX_RgXQV_eval 213
RgX_RgXQ_eval 213
RgX_Rg_add 208
RgX_Rg_add_shallow 208
RgX_Rg_div 209
RgX_Rg_divexact 209
RgX_Rg_eval_bk 209
RgX_Rg_mul 208
RgX_Rg_sub 208

RgX_Rg_translate 211
RgX_Rg_type 112
RgX_rotate_shallow 208
RgX_shift 155, 208
RgX_shift_inplace 208
RgX_shift_inplace_init 208
RgX_shift_shallow 208
RgX_splitting 143, 206
RgX_sqr 209
RgX_sqrhigh_i 212
RgX_sqrspec 210
RgX_sqr_i 209
RgX_sturmpart 252
RgX_sub 208
RgX_sylvestermatrix 233
RgX_to_F2x 171
RgX_to_Flx 171
RgX_to_FlxqX 171
RgX_to_FpX 123
RgX_to_FpXQX 128
RgX_to_FqX 128
RgX_to_nfX 308
RgX_to_RgC 225
RgX_to_RgV 226
RgX_to_ser 226
RgX_to_ser_inexact 226
RgX_type 112
RgX_type2 112
RgX_type3 112
RgX_type_code 111
RgX_type_decode 111
RgX_type_is_composite 111
RgX_unscale 211
RgX_val 208
RgX_valrem 208
RgX_valrem_inexact 208
Rg_col_ei 228
Rg_get_0 . . . . . 31, 111
Rg_get_1 111
Rg_is_FF 254
Rg_is_Fp 112
Rg_is_FpXk 133
Rg_is_FpXQ 128
Rg_is_QXk 203
Rg_nffix 331
Rg_RgC_sub 190
Rg_RgX_sub 208
Rg_to_F2 171
Rg_to_F2xq 171
Rg_to_Fl 171
Rg_to_Flxq 171
Rg_to_Fp 112, 113
Rg_to_FpXk 133
Rg_to_FpXQ 128
Rg_to_Fq 128
Rg_to_RgC 226
Rg_type 110
RM_round_maxrank 301, 325, 344
rnfabelianconjgen 331
rnfcomplete 306
rnfdisc_factored 331
rnfeltabstorel 330
rnfeltreltoabs 330
rnfeltup 330
rnfequationall 330
rnfnewprec 307
rnfnewprec_shallow 307
rnf_build_nfabs 306
rnf_COND 329
rnf_get_absdegree 303
rnf_get_alpha 304
rnf_get_degree 303
rnf_get_disc 304
rnf_get_idealdisc 304
rnf_get_index 304
rnf_get_invzk 304
rnf_get_k 304
rnf_get_map 304, 330
rnf_get_nf 303
rnf_get_nfdegree 303
rnf_get_nfpol 303
rnf_get_nfvarn 303
rnf_get_nfzk 303, 330
rnf_get_pol 303
rnf_get_polabs 304, 306
rnf_get_ramified_primes 304
rnf_get_varn 303
rnf_get_zk 304
rnf_REL 329
rnf_zkabs 306
rootmod 161
rootmod0 161
rootmod2 161
rootsof1pow 261
rootsof1powinit 261
rootsof1q_cx 261
rootsof1u_cx 260
rootsof1u_Fp 105

rootsof1_cx 260
rootsof1_Fl 105
rootsof1_Fp 105
roots_from_deg1 233
roots_to_pol 233
roots_to_pol_r1 233
roundr 89
roundr_safe 89
row vector 31
row 284
rowcopy 284
rowpermute 284
rowslice 284
rowslicepermute 284
rowsplice 284
row_i 284
row_Q_primpart 242
rtodbl 25, 223
rtor 88
rtor_lg 88
R_abs 249
R_abs_shallow 249
R_from_QR 194

S

scalarcol 228
scalarcol_shallow 233
scalarmat 228
scalarmat_s 228
scalarmat_shallow 233
scalarpol 227
scalarpol_shallow 233
scalarser 227
scalar_Flm 116
scalar_ZX 197
scalar_ZX_shallow 197
sdivsi 99
sdivsi_rem 99
sdivss_rem 98
sdomain_isincl 370
sd_breakloop 293
sd_colors 293
sd_compatible 293
sd_datadir 293
sd_debug 293
sd_debugfiles 293
sd_debugmem 293
sd_doctest 293
sd_echo 293
sd_factorlimit 293
sd_factor_add_primes 293
sd_factor_proven 293
sd_format 293
sd_graphcolormap 294
sd_graphcolors 294
sd_help 294
sd_histfile 294
sd_histsize 294
sd_intarray 295
sd_lines 294
sd_linewrap 294
sd_log 294
sd_logfile 294
sd_nbthreads 294
sd_new_galois_format 294
sd_output 294
sd_parisize 294
sd_parisizemax 294
sd_path 294
sd_plothsizes 294
sd_prettyprinter 294
sd_primelimit 294
sd_prompt 294
sd_prompt_cont 294
sd_psfile 294
sd_readline 294
sd_realbitprecision 294
sd_realprecision 294
sd_recover 294
sd_secure 294
sd_seriesprecision 294
sd_simplify 294
sd_sopath 294
sd_strictargs 294
sd_strictmatch 294
sd_string 296
sd_TeXstyle 293
sd_threadsize 294
sd_threadsizemax 295
sd_timer 294
sd_toggle 295
sd_ulong 295
secure 55
serchop0 227
serchop_i 227
sertoser 252
ser_inv 252

ser_isexactzero . . . 252
ser_normalize . . . 252
ser_unscale . . . 252
setabssign . . . 63
setalldebug . . . 39
setdefault . . . 54, 55, 293
setexpo . . . 28, 31, 63
setisclone . . . 26
setlg . . . 26, 62
setlgefint . . . 27, 62
setprecp . . . 29, 63
setrand . . . 102
setrealprecision . . . 258
setsigne . . . 27, 30, 31, 62
settyp . . . 26, 62
setunion_i . . . 238
setvalp . . . 29, 63
setvalser . . . 31, 63
setvarn . . . 24, 30, 31, 63, 233
set_avma . . . 68
set_lex . . . 291
set_sign_mod_divisor . . . 321
shallow . . . 51
shallowconcat . . . 283, 284
shallowconcat1 . . . 283
shallowcopy . . . 25, 283
shallowextract . . . 284
shallowmatconcat . . . 284
shallowmatextract . . . 284
shallowtrans . . . 283
shiftaddress . . . 68
shiftaddress_canon . . . 68
shifti . . . 90
shiftl . . . 82
shiftlr . . . 82
shiftr . . . 90
shiftr_inplace . . . 90
shift_left . . . 90
shift_right . . . 90
SIGNBITS . . . 65
signe . . . 27, 30, 31, 60
SIGNnumBITS . . . 65
SIGNSHIFT . . . 65
simplefactmod . . . 161
simplify . . . 72
simplify_shallow . . . 72
sisfundamental . . . 107
sizedigit . . . 61
SL2_inv_shallow . . . 186
smallpolred . . . 333
smallpolred2 . . . 333
SMALL_ULONG . . . 85
smith . . . 342
smithall . . . 342
smithclean . . . 343
smodis . . . 99
smodsi . . . 99
smodss . . . 99
snm_closure . . . 292
solvestep0 . . . 291
sort . . . 237
sort_factor . . . 238
split_realimag . . . 193
sprintf . . . 37
sqrfrac . . . 249
sqri . . . 93
sqrr . . . 93
sqrs . . . 93
sqrti . . . 101
sqrtnr . . . 258
sqrtnr_abs . . . 258
sqrtr . . . 258
sqrtremi . . . 101
sqrtr_abs . . . 259
sqru . . . 93
sstoQ . . . 228
stack . . . 13, 17
stackdummy . . . 70, 87
stack_calloc . . . 67
stack_calloc_align . . . 67
stack_lim . . . 22
stack_malloc . . . 67, 269
stack_malloc_align . . . 67
stack_sprintf . . . 269
stack_strcat . . . 269
stack_strdup . . . 269
stderr . . . 271
stdout . . . 36, 271
stoi . . . 25, 88
stor . . . 88
stor_lg . . . 88
Str . . . 270
strexpand . . . 270
strftime_expand . . . 273
string context . . . 77
strntoGENstr . . . 269
strprintf . . . 37
strtex . . . 270

strtoclosure 292
strtofunction 292
strtoGENstr 269, 270
strtoi 86
strtor 86
str_display_hist 57
str_init 270
str_printf 270
str_putc 270
str_puts 270
sturmpart 252
subgroups_tableset 266
subiu 96
subll 81
subllx 81
submulii 97
submuliu 97
submuliu_inplace 97
subresext 241
subui 96
subuu 96
sumalt0 291
sumdedekind 108
sumdedekind_coprime 108
sumdigitsu 107
sumdivmultexpr0 291
suminf0 291
sumnum0 291
sumnumap0 291
sumnumlagrange0 291
sumpos0 291
sunits_makecoprime 327
sunits_mod_units 332
switchin 56
switchout 271
szeta 260

T

tablemul 310
tablemulvec 310
tablemul_ei 310
tablemul_ei_ej 310
tablesearch 238
tableset_find_index 266
tablesqr 310
teichmullerinit 166
term_color 272
term_get_color 272
texe 272
theta_get_an 369
theta_get_bitprec 369
theta_get_isqrtN 370
theta_get_K 369
theta_get_m 370
theta_get_R 369
theta_get_tdom 370
threads 385
thuemorseu 82
timer 40
timer2 40
timer_delay 39, 40
timer_get 39
timer_printf 39
timer_start 39
togglesign 62
togglesign_safe 62, 183, 184
to_famat 195, 312
to_famat_shallow 195, 312
Tp_to_FF 253
trans_eval 261
trans_evalgen 261
traverseheap 73
tridiv_bound 177
tridiv_boundu 177
trivialgroup 264
trivial_fact 228
truecoef 251, 252
truecoeff 64
truedivii 97
truedivis 97
truedivsi 97
truedvmdii 99
truedvmdis 99
truedvmdsi 99
trunc2nr 90
truncr 89
trunc_safe 89
TWOPOTBITS_IN_LONG 64
typ 26, 60
TYPBITS 65
type number 26
type 14
type_name 60
TYPnumBITS 65
TYPSHIFT 65
typ_BNF 299
typ_BNR 299

typ_NF 299
t_CLOSURE 32
t_COL 31
t_COMPLEX 29
t_ELL_Fp 353
t_ELL_Fq 353, 359
t_ELL_NF 353
t_ELL_Q 353
t_ELL_Qp 353
t_ELL_Rg 354
t_ERROR 32
t_FFELT 29
t_FF_F2xq 29
t_FF_Flxq 29
t_FF_FpXQ 29
t_FRAC 29
t_INFINITY 32
t_INT 26
t_INTMOD 29
t_LIST 32
t_MAT 32
t_PADIC 29
t_POL 30
t_POLMOD 30
t_QFB 31
t_QUAD 29
t_REAL 28
t_RFRAC 31
t_SER 31
t_STR 32
t_VEC 31
t_VECSMALL 32

U

uabsdiviu_rem 99
uabsdivui_rem 99
ucoeff 64
udivuu_rem 99
uel 64
ugcd 101
ugcdiu 101
ugcdui 101
uhclassnoF_fact 335
uis2psp 107
uislucaspsp 107
uisnilpotent 107
uisprime 181
uisprimepower 107
uisprime_101 181
uisprime_661 181
uispsp 107
uissquare 175
uissquareall 175
uissquarefree 107
uissquarefree_fact 107
uis_357_power 107
uis_357_powermod 107
ulcm 101
ulogint 101
ulogintall 101
ulong 51
ULONG_MAX 64
umael 64
umael2 64
umael3 64
umael4 64
umael5 64
umodi2n 100
umodiu 99
umodsu 99
umodui 99
umuluu_le 96
umuluu_or_0 96
unegisfundamental 107
unegquadclassnoF 334
unextprime 180
unsetisclone 26
untilpari 291
uordinal 269
uposisfundamental 107
uposquadclassnoF 334
upowers 101
upowuu 101
upper_to_cx 223
uprecprime 180
uprime 180
uprimepi 180
upr_norm 316
uquadclassnoF_fact 335
usqrt 175
usqrtn 175
usumdivk_fact 108
usumdiv_fact 107
utoi 88
utoineg 88
utoipos 88
utor 88

utor_lg 88
uu32toi 88
uu32toineg 88
uutoi 89
uutoineg 89
uutoQ 228
u_chinese_coprime 162
u_forprime_arith_init 182
u_forprime_init 42, 182
u_forprime_next 42, 182
u_forprime_restrict 182
u_lval 91
u_lvalrem 91
u_lvalrem_stop 91
u_ppo 178
u_pval 91
u_pvalrem 91
u_sumdedekind_coprime 108

V

vali 89
valp 29, 60
VALPBITS 65
VALPnumBITS 65
vals 89
valser 31, 60
vandermondeinverse 191
vandermondeinverseinit 191
vandermondeinverseinit_real 192
varargs 24
varhigher 33
variable (priority) 32
variable (temporary) 34
variable (user) 33
variable number 30, 33, 76
varlower 33
varn 30, 31, 33, 60
VARNBITS 65
varncmp 32
varnmax 35
varnmin 35
VARNnumBITS 65
VARNSHIFT 65
vars_sort_inplace 74
vars_to_RgXV 74
va_list 37
vconcat 284
vec01_to_indices 320
vecan_gchar 348
vecbinomial 249
vecdiv 285
vecextract 284
vecfactoroddu 177
vecfactoroddu_i 177
vecfactorsquarefreeu 177
vecfactorsquarefreeu_coprime 177
vecfactoru 177
vecfactoru_i 177
vecindexmax 237
vecindexmin 237
vecinv 285
veclast 284
vecmodii 285
vecmoduu 285
vecmul 285
vecpermute 285
vecperm_orbits 264
vecpow 285
vecpowug 101
vecpowuu 101
vecrange 228
vecrangess 228
vecreverse 285
vecreverse_inplace 285
vecslice 284
vecslicepermute 285
vecsmall01_to_indices 320
vecsmallpermute 285
vecsmalltrunc_append 58
vecsmalltrunc_init 58
vecsmall_append 287
vecsmall_coincidence 287
vecsmall_concat 287
vecsmall_copy 286
vecsmall_counting_indexsort 287
vecsmall_counting_sort 287
vecsmall_counting_uniq 287
vecsmall_duplicate 287
vecsmall_duplicate_sorted 287
vecsmall_ei 228
vecsmall_indexmax 287
vecsmall_indexmin 287
vecsmall_indexsort 286
vecsmall_is1to1 287
vecsmall_isconst 287
vecsmall_isin 287
vecsmall_lengthen 286

vecsmall_lexcmp 287
vecsmall_max 286
vecsmall_min 287
vecsmall_pack 287
vecsmall_prefixcmp 287
vecsmall_prepend 287
vecsmall_prod 287
vecsmall_reverse 286
vecsmall_shorten 286
vecsmall_sort 286
vecsmall_to_col 286
vecsmall_to_vec 286
vecsmall_to_vec_inplace 286
vecsmall_uniq 287
vecsmall_uniq_sorted 287
vecsort 237
vecsplice 284
vecsqr 285
vecsquarefreeu 177
vectrunc_append 58
vectrunc_append_batch 58
vectrunc_init 58
vecvecsmall_indexsort 288
vecvecsmall_max 288
vecvecsmall_search 288
vecvecsmall_sort 287
vecvecsmall_sort_inplace 287
vecvecsmall_sort_shallow 287
vecvecsmall_sort_uniq 288
veczeta 260
vec_append 286
vec_ei 228
vec_equiv 286
vec_insert 286
vec_is1to1 286
vec_isconst 285
vec_lengthen 286
vec_prepend 286
vec_Q_primpart 242
vec_reduce 286
vec_setconst 285
vec_shorten 286
vec_to_vecsmall 286
Vgaeasytheta 371

W

walltimer_delay 39
walltimer_get 39
walltimer_start 39
warner 38
warnfile 38
warnmem 38
warnprec 38
whilepari 291
writebin 36, 272

Z

Z2_sqrt 166
ZabM_indexrank 310
ZabM_inv 310
ZabM_inv_ratlift 311
ZabM_ker 310
ZabM_pseudoinv 311
zbrent0 291
zCs_to_ZC 196
ZC_add 183
ZC_copy 183
ZC_divexactu 183
ZC_hnfdivrem 345
ZC_hnfrem 345
ZC_is_ei 186
ZC_lincomb 183
ZC_lincomb1_inplace 183
ZC_lincomb1_inplace_i 183
ZC_modhnf 340
ZC_neg 183
ZC_nfval 319
ZC_nfvalrem 320
ZC_prdvd 320
ZC_Q_mul 185
ZC_reducemodlll 345
ZC_reducemodmatrix 345
ZC_sub 183
zc_to_ZC 173
ZC_union_shallow 184
ZC_ZV_mul 183
ZC_Z_add 183
ZC_Z_div 183
ZC_Z_divexact 183
ZC_z_mul 173
ZC_Z_mul 183
ZC_Z_sub 183
zerocol 228
zeromat 228
zeromatcopy 228
zeropadic 227

zeropadic_shallow 233
zeropol 228
zeroser 227
zerovec 228
zerovec_block 228
zero_F2m 120
zero_F2m_copy 120
zero_F2v 120
zero_F2x 155
zero_F3m_copy 121
zero_F3v 121
zero_Flm 118
zero_Flm_copy 118
zero_Flv 118
zero_Flx 142
zero_FlxC 145
zero_FlxM 145
zero_FlxM_copy 145
zero_zm 188
zero_zv 188
zero_zx 205
ZGCs_add 195
ZGC_G_mul 195
ZGC_G_mul_inplace 195
ZGC_Z_mul 195
ZG_add 195
ZG_G_mul 195
ZG_mul 195
ZG_neg 195
ZG_normalize 195
ZG_sub 195
ZG_Z_mul 195
Zideallog 348
zidealstar 332
zidealstarinit 332
zidealstarinitgen 332
zkchinese 315
zkchinese1 315
zkchineseinit 315
zkC_multable_mul 309
zkmodprinit 319
zkmultable_capZ 310
zkmultable_inv 310
zk_inv 309, 310
zk_multable 309, 314
zk_scalar_or_multable 310, 316
zk_to_Fq 319
zk_to_Fq_init 319
zlm_echelon 169
ZlM_gauss 170
zlxX_translate1 149
zlx_translate1 141
ZM2_mul 184
ZM2_sqr 185
ZMrow_equal0 185
ZMrow_ZC_mul 185
zMs_compress 196
zMs_to_ZM 196
zMs_ZC_mul 196
ZMV_prod 185
ZMV_producttree 163
ZMV_to_FlmV 189
ZMV_to_zmV 189
zmV_to_ZMV 189
ZM_add 184
ZM_charpoly 186
ZM_copy 184
zm_copy 188
ZM_det 185
ZM_detmult 185
ZM_det_triangular 186
ZM_diag_mul 185
ZM_divexactu 185
ZM_equal 184
ZM_equal0 184
ZM_famat_limit 312
ZM_gauss 186
ZM_hnf 338, 342
ZM_hnfall 338, 339, 340, 342, 343
ZM_hnfall_i 339
ZM_hnfcenter 340
ZM_hnfdivrem 345
ZM_hnfintersectmod 338
ZM_hnflll 340
ZM_hnfmod 338, 342
ZM_hnfmodall 339
ZM_hnfmodall_i 339
ZM_hnfmodid 338, 339, 342
ZM_hnfmodprime 338
ZM_hnfperm 339
ZM_hnfrem 345
ZM_hnf_knapsack 339
ZM_imagecompl 186
ZM_incremental_CRT 162
ZM_indeximage 186
ZM_indexrank 186
ZM_init_CRT 162
ZM_inv 186

ZM_inv_ratlift 186
ZM_isdiagonal 186
ZM_ishnf 186
ZM_isidentity 186
ZM_isscalar 186
ZM_ker 186
ZM_lll 343, 344
ZM_lll_norms 344
ZM_max_expi 186
ZM_max_lg 186
ZM_merge_factor 238
ZM_modhnf 340
ZM_mul 184
zm_mul 188
ZM_multosym 185
ZM_mul_diag 185
ZM_neg 184
ZM_nm_mul 173
ZM_nv_mod_tree 163
ZM_permanent 185
zm_permanent 188
ZM_pow 185
ZM_powu 185
ZM_pseudoinv 186
ZM_Q_mul 185
ZM_rank 186
ZM_reducemodlll 345
ZM_reducemodmatrix 345
zm_row 188
ZM_snf 340
ZM_snfall 340
ZM_snfall_i 340
ZM_snfclean 341
ZM_snf_group 341
ZM_sqr 184
ZM_sub 184
ZM_supnorm 186, 245
ZM_togglesign 184
ZM_to_F2m 120
ZM_to_F3m 122
ZM_to_Flm 172
zm_to_Flm 173
ZM_to_zm 172
zm_to_ZM 173
zm_to_zxV 174
ZM_trace 186
ZM_transmul 185
ZM_transmultosym 185
zm_transpose 188
ZM_zc_mul 173
ZM_ZC_mul 185
zm_zc_mul 187
ZM_zm_mul 173
ZM_ZV_mod 185
ZM_ZX_mul 185
ZM_Z_div 185
ZM_Z_divexact 185
ZM_Z_mul 185
zncharcheck 347
zncharconj 347
znchardiv 347
znchareval 347
zncharker 347
zncharmul 347
zncharorder 347
zncharpow 347
znchar_quad 348
znconreyfromchar 347
znconreyfromchar_normalized 347
znconreylog_normalize 348
znconrey_check 347
znconrey_normalized 347
znstar_get_conreycyc 306
znstar_get_conreygen 306
znstar_get_cyc 305
znstar_get_faN 305
znstar_get_gen 305
znstar_get_N 305
znstar_get_no 305
znstar_get_pe 305
znstar_get_U 306
znstar_get_Ui 306
Zn_ispower 105
Zn_issquare 105
Zn_quad_roots 105
Zn_sqrt 104
Zn_sqrtall 104
ZpMs_ZpCs_solve 197
ZpM_echelon 169
ZpM_invlift 166
ZpM_ZpC_gauss 166
ZpXQM_prodFrobenius 168
ZpXQXQ_div 169
ZpXQXQ_inv 169
ZpXQXQ_invlift 169
ZpXQX_digits 169
ZpXQX_divrem 169
ZpXQX_liftfact 168

ZpXQX_liftroot 168, 169
ZpXQX_liftroots 169
ZpXQX_liftroot_vald 169
ZpXQX_roots 169
ZpXQX_ZpXQXQ_liftroot 169
ZpXQ_div 168
ZpXQ_inv 168
ZpXQ_invlift 168
ZpXQ_log 168
ZpXQ_sqrt 168
ZpXQ_sqrtnlift 168
ZpX_disc_val 167
ZpX_Frobenius 167
ZpX_gcd 167
ZpX_liftfact 167
ZpX_liftroot 166, 168
ZpX_liftroots 167
ZpX_monic_factor 167
ZpX_primedec 167
ZpX_reduced_resultant 167
ZpX_reduced_resultant_fast 167
ZpX_resultant_val 167
ZpX_roots 166
ZpX_ZpXQ_liftroot 168
ZpX_ZpXQ_liftroot_ea 168
Zp_div 166
Zp_exp 166
Zp_inv 166
Zp_invlift 166
Zp_issquare 105
Zp_log 166
Zp_sqrt 166
Zp_sqrtlift 166
Zp_sqrtnlift 166
Zp_teichmuller 166
ZqX_liftfact 169
ZqX_liftroot 169
ZqX_roots 169
ZqX_ZqXQ_liftroot 169
Zq_sqrtnlift 168
zvV_equal 188
ZVV_nv_mod_tree 163
ZVV_to_FlvV 172
zv_abs 187
ZV_abscmp 182
ZV_allpnqn 102
zv_canon_inplace 187
ZV_cba 178
ZV_cba_extend 178
ZV_chinese 164
ZV_chinesetree 165
ZV_chinese_center 164
ZV_chinese_tree 165
ZV_cmp 182, 239
zv_cmp0 188
ZV_content 183
zv_content 188
zv_copy 188
zv_cyc_minimal 347
zv_cyc_minimize 347
zv_diagonal 188
ZV_dotproduct 183
zv_dotproduct 188
ZV_dotsquare 183
ZV_dvd 184
ZV_equal 182
zv_equal 188
ZV_equal0 182
zv_equal0 188
ZV_extgcd 102, 183
ZV_indexsort 184
ZV_isscalar 192
ZV_lcm 102
ZV_lval 92
ZV_lvalrem 92
ZV_max_expi 184
ZV_max_lg 184
zv_neg 187
ZV_neg_inplace 183
zv_neg_inplace 187
ZV_nv_mod_tree 163
ZV_prod 184
zv_prod 188
ZV_producttree 163, 165
zv_prod_Z 188
ZV_pval 92
ZV_pvalrem 92
ZV_search 184
zv_search 188
ZV_snfall 340
ZV_snfclean 341
ZV_snf_gcd 102, 341
ZV_snf_group 341
ZV_snf_rank 342
zv_snf_rank 342
ZV_snf_rank_u 342
ZV_snf_trunc 341
ZV_sort 184

ZV_sort_inplace . . . 184
ZV_sort_shallow . . . 184
ZV_sort_uniq . . . 184
ZV_sort_uniq_shallow . . . 184
ZV_sum . . . 184
zv_sum . . . 188
zv_sumpart . . . 188
ZV_togglesign . . . 183
ZV_to_F2v . . . 120
ZV_to_F3v . . . 121
ZV_to_Flv . . . 171
zv_to_Flv . . . 173
ZV_to_nv . . . 172
ZV_to_zv . . . 172
zv_to_ZV . . . 173
zv_to_zx . . . 173
ZV_union_shallow . . . 184
ZV_zc_mul . . . 173
ZV_zMs_mul . . . 196
zv_ZM_mul . . . 173
ZV_ZM_mul . . . 185
ZV_ZV_mod . . . 183
ZV_Z_dvd . . . 92
zv_z_mul . . . 188
ZXC_nv_mod_tree . . . 163
ZXk_divexact . . . 203
ZXk_divides . . . 203
ZXk_gcd . . . 203
ZXM_incremental_CRT . . . 163
ZXM_init_CRT . . . 163
ZXM_nv_mod_tree . . . 164
ZXn_mul . . . 202
ZXn_sqr . . . 202
ZXQM_mul . . . 202
ZXQM_sqr . . . 202
ZXQXQ_minpoly . . . 202
ZXQX_dvd . . . 210
ZXQX_gcd . . . 202
ZXQX_mul . . . 202
ZXQX_sqr . . . 202
ZXQX_ZXQ_mul . . . 202
ZXQ_charpoly . . . 202
ZXQ_minpoly . . . 202
ZXQ_mul . . . 201
ZXQ_powers . . . 202
ZXQ_powu . . . 202
ZXQ_sqr . . . 201
ZXT_remi2n . . . 201
ZXT_to_FlxT . . . 171
ZXV_dotproduct . . . 201
ZXV_equal . . . 201
ZXV_pval . . . 92
ZXV_pvalrem . . . 92
ZXV_remi2n . . . 201
ZXV_to_FlxV . . . 171
ZXV_ZX_fromdigits . . . 199
ZXV_Z_mul . . . 201
ZXXT_to_FlxXT . . . 172
ZXXV_to_FlxXV . . . 172
ZXX_evalx0 . . . 203
ZXX_max_lg . . . 203
ZXX_mul_Kronecker . . . 203
ZXX_nv_mod_tree . . . 164
ZXX_pval . . . 92
ZXX_pvalrem . . . 92
ZXX_Q_mul . . . 203
ZXX_renormalize . . . 203
ZXX_resultant . . . 201
ZXX_shifti . . . 203
ZXX_sqr_Kronecker . . . 203
ZXX_to_F2xX . . . 157
ZXX_to_FlxX . . . 171
zxX_to_FlxX . . . 172
zxX_to_Kronecker . . . 150
ZXX_Z_add_shallow . . . 203
ZXX_Z_divexact . . . 203
ZXX_Z_mul . . . 203
ZX_add . . . 198
ZX_affine . . . 200
ZX_composedsum . . . 201
ZX_compositum . . . 201
ZX_content . . . 199
ZX_copy . . . 197
ZX_deflate_max . . . 199
ZX_deflate_order . . . 199
ZX_deriv . . . 200
ZX_digits . . . 199
ZX_disc . . . 200
ZX_divuexact . . . 198
ZX_div_by_X_1 . . . 198
ZX_equal . . . 198, 201
ZX_equal1 . . . 198
ZX_eval1 . . . 200
ZX_factor . . . 200
ZX_gcd . . . 199
ZX_gcd_all . . . 199
ZX_graeffe . . . 200
ZX_incremental_CRT . . . 163

ZX_init_CRT 163
ZX_is_irred 200
ZX_is_monic 198
ZX_is_squarefree 200
ZX_lval 92
zx_lval 205
ZX_lvalrem 92
ZX_max_lg 197
ZX_mod_Xnm1 198
ZX_mul 198, 203
ZX_mulspec 198
ZX_mulu 198
ZX_neg 198
ZX_nf_roots 301
ZX_nv_mod_tree 163
ZX_primitive_to_monic 199
ZX_pval 92
ZX_pvalrem 92
ZX_Q_mul 199
ZX_Q_normalize 199, 322
ZX_radical 199
ZX_realroots 201
ZX_realroots_irred 201
ZX_rem 198
ZX_remi2n 198
ZX_renormalize 198
zx_renormalize 205
ZX_rescale 199
ZX_rescale2n 199
ZX_rescale_lt 199
ZX_resultant 200
zx_shift 205
ZX_shifti 198
ZX_sqr 198, 203
ZX_sqrspec 198
ZX_squff 200
ZX_sturm 201
ZX_sturmpart 201, 252
ZX_sturm_irred 201
ZX_sub 198
ZX_to_F2x 154
ZX_to_Flx 171
zx_to_Flx 172
ZX_to_monic 199
zx_to_zv 173
zx_to_ZX 172
ZX_unscale 200
ZX_unscale2n 200
ZX_unscale_div 200
ZX_unscale_divpow 200
ZX_Uspensky 200
ZX_val 199
ZX_valrem 199
ZX_Zp_root 166
ZX_ZXY_resultant 201
ZX_ZXY_rnfequation 201
ZX_Z_add 198
ZX_Z_add_shallow 198
ZX_Z_divexact 198
zx_z_divexact 205
ZX_Z_eval 200
ZX_Z_mul 198
ZX_Z_normalize 199
ZX_Z_sub 198
ZX_Z_translate 199
ZX_z_unscale 200
Z_cba 178
Z_chinese 161
Z_chinese_all 161
Z_chinese_coprime 161
Z_chinese_post 162
Z_chinese_pre 162
Z_content 242
Z_ECM 177
Z_factor 175, 177, 178
Z_factor_limit 176, 178
Z_factor_listP 176
Z_factor_until 176
Z_FF_div 255
Z_has_prime_3mod4 176
Z_incremental_CRT 162
Z_init_CRT 162
Z_isanypower 175, 178
Z_isfundamental 180
Z_ispow2 175
Z_ispower 175
Z_ispowerall 175
Z_issmooth 176
Z_issmooth_fact 176
Z_issquare 174
Z_issquareall 175
Z_issquarefree 180
Z_issquarefree_fact 180
Z_lsmoothen 176
Z_lval 91
z_lval 91
Z_lvalrem 91
z_lvalrem 91

Z_lvalrem_stop 91
Z_nv_mod 163
Z_pollardbrent 177
Z_ppgle 178
Z_ppio 178
Z_ppo 178
Z_pval 91
z_pval 91
Z_pvalrem 91
z_pvalrem 91
Z_smoothen 176
Z_to_F2x 154
Z_to_famat 312
Z_to_Flx 173
Z_to_FpX 124
Z_to_perm 264
Z_ZC_sub 183
Z_ZV_mod 163
Z_ZV_mod_tree 163
Z_ZX_sub 198

.

_evalexpo 62
_evallg 61
_evalprecp 62
_evalvalp 62
_evalvalser 62